Explicit Vinogradov–Korobov regions for Dirichlet L-functions

Kenan

July 22, 2026

Abstract

Put $V(t) = (\log t)^{2/3}(\log\log t)^{1/3}$. We prove that, for every $q \geq 3$, every Dirichlet character $\chi \pmod q$, and every real $|t| \geq 10$,

$$L(\sigma + it, \chi) \neq 0 \quad \text{if} \quad \sigma \geq 1 - \frac{1}{10.295195 \log q + 61.272145 V(|t|)}.$$

This improves both coefficients in Khale's published global pair $(10.5, 61.5)$. We also construct an exact rational sum-of-squares trigonometric polynomial of degree 46. It gives an absolute effectively computable Y such that, for $|t| \geq Y$, the corresponding pair is $(9.93082, 48.07157)$, while the zeta coefficient is 48.07157. The threshold Y is not numerically evaluated. For the explicit Goldbach summatory framework of Bhowmik–Ernvall-Hytönen–Palojärvi, the global pair lowers the formal logarithmic threshold to $64923900841612.7437\ldots$, decreases the zero-region input, and widens the associated at-most-one-zero region. We show, however, that the printed uniform coefficients 5.805 and 7.246 do not follow from that specialization, and we correct a factor of two in its optional Lambert-W inversion.

Keywords. Dirichlet L-function; zero-free region; Vinogradov–Korobov method; trigonometric polynomial; interval arithmetic; Goldbach problem.

2020 Mathematics Subject Classification. Primary 11M06, 11M26; Secondary 11L15, 11P32, 65G20.

1 Introduction

The Vinogradov–Korobov zero-free region is the largest classical zero-free region near $s = 1$ available uniformly at large height. For a Dirichlet character $\chi \pmod q$, its explicit form naturally has two scales,

$$c_1 \log q + c_2(\log|t|)^{2/3}(\log\log|t|)^{1/3}.$$

Khale established the first explicit all-character theorem of this form and proved the global pair $(10.5, 61.5)$ for $|t| \geq 10$, together with the effective large-height pair $(10.1, 49.13)$ [5]. His proof combines a Hurwitz-zeta growth estimate, a nonnegative trigonometric polynomial, a local zero detector, and a finite-height seed.

There are two distinct optimization problems. The first is global: every height from 10 onward must be covered, and the finite-height classical region becomes decisive. The second is asymptotic: fixed-degree lower-order terms may be absorbed above an effectively computable height, so the main conductor and height ratios can be optimized directly. These problems have different best constants, and conflating them would replace an explicit theorem by an unevaluated assertion.

Our first main theorem is global.

Theorem 1.1 (Global region)**.** *Let $q \geq 3$, let $\chi \pmod{q}$ be any Dirichlet character, and let $|t| \geq 10$. Then*

$$\boxed{L(\sigma+it,\chi) \neq 0 \quad \textit{for} \quad \sigma \geq 1 - \frac{1}{10.295195\log q + 61.272145V(|t|)}}.$$

The second theorem is stronger at sufficiently large height but does not give a numerical value for its starting point.

Theorem 1.2 (Effective large-height regions)**.** *There is an absolute effectively computable $Y > 0$ such that, for every real $|t| \geq Y$,*

$$\zeta(\sigma+it) \neq 0 \quad \textit{for} \quad \sigma \geq 1 - \frac{1}{48.07157V(|t|)}.$$

For the same Y, every $q \geq 3$ and every character $\chi \pmod{q}$ satisfy

$$\boxed{L(\sigma+it,\chi) \neq 0 \quad \textit{for} \quad \sigma \geq 1 - \frac{1}{9.93082\log q + 48.07157V(|t|)}}.$$

The asymptotic theorem is driven by an exact rational sum-of-squares polynomial rather than by printing more digits from a previously used polynomial. The certificate is a 47-by-47 integer Cholesky factor. Exact multiplication proves both global nonnegativity and positivity of every cosine coefficient. Directed interval arithmetic is then used only for transcendental evaluations and strict comparisons.

The global theorem uses a different mechanism. At the fixed height $\log T_0 = 1943.408$, we split by the conductor d of the primitive character inducing χ. The principal branch is controlled by a global zeta region, conductors $3 \leq d \leq 400000$ by Kadiri's 5.60 theorem [4], and conductors $d \geq 400001$ by McCurley's region [6]. A fixed-threshold form of Khale's argument then propagates this seed to every larger height. Joint optimization of the large-conductor bridge and the propagation envelope produces the displayed pair.

The final part of the paper records exactly what these constants change in the explicit Goldbach summatory estimates of Bhowmik–Ernvall-Hytönen–Palojärvi [2]. The parameter maps improve, but two printed fixed numerical coefficients are not justified uniformly in q by the source specialization. We also give the corrected Lambert-W condition needed to exclude the possible exceptional zero. No assertion in this paper concerns a possible real exceptional zero at height zero, and none of the results implies the Riemann hypothesis or the generalized Riemann hypothesis.

2 Notation and analytic inputs

Throughout,

$$V(t) = (\log t)^{2/3}(\log\log t)^{1/3} \qquad (t > 1).$$

Displayed terminating decimals in certificate formulas are interpreted as exact rational numbers.

2.1 The exact zeta and Hurwitz-zeta growth pair

Bellotti proves an exponential-sum estimate with the displayed constants

$$C = 8.7979, \qquad D = 132.94357$$

and maps these constants to zeta and Hurwitz-zeta growth estimates [1]. Retaining the exact displayed values gives the following slightly sharper pair.

Proposition 2.1. *Set*

$$A_* = \frac{C + 1 + 10^{-80}}{(108 \log 10)^{2/3}} 1.569\, CD^{1/3}, \qquad B_* = \frac{2}{9}\sqrt{3D}.$$

For $|t| \geq 3$, $1/2 \leq \sigma \leq 1$, and $0 < u \leq 1$,

$$|\zeta(\sigma + it)|, \quad \left|\zeta(\sigma + it, u) - u^{-\sigma - it}\right| \leq A_* \, |t|^{B_*(1-\sigma)^{3/2}} \log^{2/3} |t| .$$

Moreover,

$$A_* = 70.69940017846693326999\ldots < 70.699401,$$
$$B_* = 4.43794363457938745563\ldots < 4.4379437.$$

Proof. Bellotti's high-range coefficient is

$$a(t) = \frac{C + 1 + 10^{-80}}{(\log t)^{2/3}} + 1.569\, CD^{1/3}.$$

It decreases for $t \geq 10^{108}$, so its endpoint is A_*. On the two remaining source ranges the coefficients are respectively 70.6199 and 58.1, both with exponent 4. They are dominated by A_*, B_*. The numerical enclosures follow by directed evaluation of the displayed formulas, and conjugation handles negative t. □

2.2 Global zeta input

We use the following global zeta theorem only on the principal-character branch.

Proposition 2.2. *For every real $|t| \geq 3$,*

$$\zeta(\sigma + it) \neq 0 \quad \textit{if} \quad \sigma > 1 - \frac{1}{51.339999 V(|t|)}.$$

This is a corrected computer-assisted implementation of Yang's global Vinogradov–Korobov architecture [9]. Its certificate uses an exact positive degree-240 trigonometric polynomial, a complete Jensen-semicycle bound, corrected density integration, a monotone fixed-point step, and overlapping lower-height splices. In the proof of theorem 1.1, only the theorem statement is used.

2.3 A fixed-threshold form of Khale's propagation

Put

$$L_0 = 1943.408, \qquad T_0 = e^{L_0},$$

and define

$$a = 10.082 + \frac{1.607}{\log L_0}, \qquad b = 9.791.$$

The following is the form of Khale's high-height argument needed below.

Lemma 2.3 (Fixed-threshold propagation)**.** *There is a constant M_1 satisfying*

$$\frac{B_*^{2/3}}{M_1} < 61.0088508661$$

with the following property. Fix $q \geq 3$ and $0 < M_0 \leq M_1$. If every character modulo q is zero-free on $T_0 - 1 \leq t \leq T_0$ in the region with denominator

$$D_{q,M_0}(t) = a \log q + b \log \log q + \frac{B_*^{2/3}}{M_0} V(t),$$

then the same region is zero-free for every $t \geq T_0$.

Proof. Khale's source detector has a parameter M defined by a first counterexample. His estimates require only

$$M \leq M_0 \leq M_1 \leq 0.055071.$$

Repeating the contradiction argument with

$$\delta = \min \left\{ \frac{10^{-100}}{L_0}, \frac{M_0 - M}{2M} \right\}$$

leaves every analytic estimate unchanged and yields $M > M_1 \geq M_0$, contrary to $M < M_0$. The numerical bound for M_1 follows from proposition 2.1 and the explicit maximum $X(t) < 5.531$ in Khale's auxiliary function for $t \geq T_0$. $\square$

3 The exact degree-46 detector

3.1 An integer sum-of-squares certificate

Let $\mathcal{L} = (\ell_{ij})_{0 \leq j \leq i \leq 46}$ be the integer lower-triangular matrix supplied in the supplementary certificate `degree46-candidate.json`. Define

$$\mathcal{Q} = \mathcal{L}\mathcal{L}^{\mathsf{T}}, \qquad D_0 = \operatorname{tr} \mathcal{Q}, \qquad v(x) = (1, e^{ix}, \ldots, e^{46ix})^{\mathsf{T}},$$

and

$$P(x) = \frac{v(x)^* \mathcal{Q} v(x)}{D_0}.$$

Proposition 3.1. *The polynomial P has an exact expansion*

$$P(x) = 1 + \sum_{k=1}^{46} b_k \cos(kx)$$

such that $P(x) \geq 0$ for every real x, $b_k > 0$ for $1 \leq k \leq 46$, and

$$b_1 = 1.7470750000000740681514683620\ldots,$$
$$b := \sum_{k=1}^{46} b_k = 3.5742761313277308231079554746\ldots.$$

Proof. The identity

$$P(x) = \frac{\|\mathcal{L}^{\mathsf{T}} v(x)\|_2^2}{D_0}$$

proves nonnegativity without numerical eigenvalue testing. Exact integer correlations give

$$b_k = \frac{2}{D_0} \sum_{i=0}^{46-k} \mathcal{Q}_{i,i+k}.$$

The supplementary checker evaluates these 46 rational numbers, proves each numerator positive, and computes the displayed decimal enclosures by directed division. □

3.2 The limiting ratios

Let $\theta \in (0, \pi/2)$ be the required solution of

$$\sin^2\theta = b_1(1 - \theta\cot\theta), \qquad \alpha = \cos^2\theta.$$

Retaining Khale's positive parity correction, set

$$w(0) = \sec^2\theta(\theta\tan\theta + 3\theta\cot\theta - 3),$$

$$\bar{\alpha} = \alpha - \frac{b}{w(0)} e^{-1937}.$$

Finally define

$$H_P(E) = \frac{b+1}{3E} + \frac{b}{2}E^{1/2}, \qquad E_P = \left(\frac{4}{3}\left(1 + \frac{1}{b}\right)\right)^{2/3}.$$

Proposition 3.2. *Directed interval evaluation gives*

$$\theta = 1.1327007653845105423211131434\ldots,$$
$$\alpha = 0.1799589174582943284638858094\ldots,$$
$$\frac{b}{2\bar{\alpha}} = 9.9308113813144967884694271604\ldots < 9.93082,$$
$$\frac{B_*^{2/3} H_P(E_P)}{\bar{\alpha}} = 48.07155656004025\ldots < 48.07157.$$

The parity subtraction is positive and less than 10^{-841}.

Proof. The smoothing equation has opposite signs at the endpoints of a rational interval of width 10^{-28}, and its derivative has fixed sign there. Bisection in real balls isolates θ. All remaining quantities are then evaluated outward from exact rational inputs. The two target inequalities are checked by cross multiplication, not by rounded decimal comparison. □

4 Proof of the effective large-height theorem

Mossinghoff–Trudgian–Yang formulate the zeta detector for every fixed nonnegative trigonometric polynomial with nonnegative coefficients and $b_1 > b_0$ [7]. Bellotti's improvement replaces the exponential-sum exponent by B_*. For the polynomial of proposition 3.1, the limiting zeta denominator is exactly

$$\frac{B_*^{2/3} H_P(E_P)}{\alpha}.$$

The strict reserve in proposition 3.2 absorbs all effective remainder terms above an absolute computable height Y_ζ, giving the zeta part of theorem 1.2.

For Dirichlet L-functions, Khale's detector extends termwise to any fixed degree. The conductor main term is

$$\frac{b}{2} \log q,$$

while the contributions of imprimitive powers are $O_P(\log\log q)$, and all remaining conductor, zero-count, transform, and outer-line terms are lower order on the $\log q + V(t)$ scale. Consequently the limiting conductor coefficient is $b/(2\bar{\alpha})$, and the limiting height coefficient is the second ratio in proposition 3.2.

The seed must be simultaneous for all characters modulo q, since the detector uses powers χ^j. Thorner and Zaman prove the effective log-free family estimate

$$N_q(\sigma, T) = \sum_{\psi \pmod q} N_\psi(\sigma, T) \ll_\varepsilon (qT)^{(12/5+\varepsilon)(1-\sigma)}$$

[8]. At a sufficiently large absolute height this produces a unit interval common to every character modulo q and hence closed under all power maps.

To propagate the common seed, take the infimum over zeros of all characters modulo q of

$$Z_Q(\beta, \gamma) = \frac{(1-\beta) B_*^{2/3} V(\gamma)}{1 - Q(1-\beta)\log q}.$$

The fixed-polynomial detector excludes an interior minimizer at the target ratios. Primitive conductors divide q; the extra Euler factors of imprimitive characters have zeros on $\operatorname{Re} s = 0$. The new zeta region excludes the principal-character branch at coefficient 48.07157. Thus the common seed propagates to every larger height. Taking $Y = \max(Y_\zeta, Y_D)$, and using conjugation for negative ordinates, proves theorem 1.2.

Remark 4.1. *The argument proves that Y is effectively computable, but this paper does not evaluate it. Therefore theorem 1.2 cannot replace a finite-height table or a global explicit cutoff.*

5 The finite-height conductor split

Let $\chi \pmod q$ be induced by the primitive character χ^* of conductor $d \mid q$. Then

$$L(s, \chi) = L(s, \chi^*) \prod_{\substack{p \mid q \\ p \nmid d}} (1 - \chi^*(p) p^{-s}).$$

Every zero of an additional Euler factor lies on $\operatorname{Re} s = 0$, outside the regions considered here.

Fix

$$Q = 10.295195, \qquad C_0 = 61.272145,$$

and write

$$T_q(t) = Q \log q + C_0 V(t).$$

We first prove zero-freeness for $10 \le |t| \le T_0$.

Proposition 5.1. *For every $q \ge 3$ and every character modulo q, the denominator $T_q(t)$ gives a zero-free region throughout $10 \le |t| \le T_0$.*

Proof. There are three primitive-conductor branches.

Principal branch, $d = 1$. The primitive factor is $\zeta(s)$. Since

$$Q \log q + (C_0 - 51.339999)V(t) > 0,$$

the target region is contained in proposition 2.2.

Kadiri branch, $3 \le d \le 400000$. Kadiri's theorem has denominator

$$5.60 \log(d \max(1, |t|)).$$

As a function of $L = \log |t|$,

$$L^{2/3}(\log L)^{1/3}$$

is strictly concave on the relevant interval. The comparison therefore reduces to $L = \log 10$ and $L = L_0$. Directed evaluation gives the denominator margins

$$92.8335854352\ldots, \qquad 7859.6348469689\ldots.$$

McCurley branch, $d \ge 400001$. McCurley's all-modulus region has denominator

$$9.645908801 \log \max(d, d\,|t|\,, 10)$$

and at most one possible zero, which is real. At nonzero ordinate it is therefore zero-free. The worst conductor is $d = 400001$, and concavity again reduces the check to the two height endpoints. The margins are

$$86.7346242519\ldots, \qquad 0.00040394462695 09\ldots.$$

Equivalently, the limiting height coefficient is

$$61.2721436790941914\ldots < C_0.$$

These branches cover every primitive conductor. Conjugation gives negative ordinates. □

6 Fixed-threshold propagation and the global pair

For $x = \log q$, define

$$r(x) = (Q - a)x - b \log x$$

and

$$C_q = C_0 + \frac{\min(r(x), 0)}{V(T_0)}, \qquad M_0(q) = \frac{B_*^{2/3}}{C_q}.$$

Proposition 6.1. *For every $q \geq 3$,*

$$C_q \geq 61.0090204549319203\ldots > 61.009,$$

and hence $M_0(q) \leq M_1$.

Proof. The function r is strictly convex and has its unique minimum at

$$x_* = \frac{b}{Q-a} = 10080.3539410749\ldots.$$

Substitution gives the displayed global floor. The last implication follows from

$$\frac{B_*^{2/3}}{M_1} < 61.0088508661 < 61.009.$$

□

Put

$$D_q(t) = a\log q + b\log\log q + C_q V(t).$$

If $r(\log q) < 0$, then

$$T_q(t) - D_q(t) = r(\log q)\left(1 - \frac{V(t)}{V(T_0)}\right).$$

This is nonpositive below T_0, so proposition 5.1 supplies the required D_q seed; it is nonnegative above T_0, so the propagated D_q region contains the target.

If $r(\log q) \geq 0$, then

$$r(\log 3) < 0, \qquad r(200) < 0.$$

Convexity forces $\log q > 200$. McCurley's theorem directly supplies the fixed-threshold seed there, with denominator margin

$$136.0920701585\ldots.$$

Applying lemma 2.3 in both cases proves theorem 1.1.

6.1 The fixed-architecture frontier

At the chosen L_0, the two active coefficient constraints are

$$B(Q) = \frac{9.645908801(\log 400001 + L_0) - Q\log 400001}{V(T_0)}$$

and

$$E(Q) = 61.009 + \frac{9.791}{V(T_0)}\left(\log\frac{9.791}{Q-a} - 1\right).$$

The first is the large-conductor bridge; the second is the propagation envelope.

Proposition 6.2. *The functions B and E have a unique crossing on the relevant interval, with*

$$Q_* = 10.2951944189592868\ldots,$$
$$C_* = 61.2721437036028819\ldots.$$

The rational pair $(10.295195, 61.272145)$ lies strictly above both constraints.

Proof. On the relevant interval,

$$\frac{\mathrm{d}}{\mathrm{d}Q}(B-E) = \frac{-\log 400001 + 9.791/(Q-a)}{V(T_0)} > 0.$$

Directed bisection brackets the crossing in an interval of width 10^{-12}. Substitution of the rational target gives positive bridge and envelope reserves. □

Remark 6.3. *Proposition 6.2 is an optimization statement only for the fixed threshold $L_0 = 1943.408$ and the particular Kadiri–McCurley–Khale architecture. It is not a global optimality claim for Dirichlet L-function zero-free regions.*

7 Goldbach propagation and its limits

We now apply theorem 1.1 to the parameterized framework in [2]. Write

$$\eta_q(t) = \frac{1}{c_1 \log q + c_2 V(t)}, \qquad B_q^*(t) = \min\{B_q, 1-\eta_q(t)\}.$$

The published substitution uses $(c_1, c_2) = (10.5, 61.5)$, while theorem 1.1 permits

$$(c_1, c_2) = (10.295195, 61.272145).$$

7.1 The formal threshold and zero-region ordering

For $x_0 = e^e$, the displayed threshold map in the source is

$$H(c_1, c_2) = \exp\left(-4W_{-1}\left(-\left[4 \cdot 5^{3/4}(3.5c_1e^{-2/3} + c_2)^{3/4}\right]^{-1}\right)\right).$$

Proposition 7.1. *The global pair gives*

$$H(10.295195, 61.272145) = 64923900841612.7437791589\ldots.$$

Thus the least sufficient integer ceiling in this formal map is

$$64923900841613.$$

It is smaller than the published ceiling by 1679279589013. *Moreover, $\eta_q(t)$ increases and $B_q^*(t)$ weakly decreases for every admissible q, t.*

Proof. The argument of W_{-1} lies in $(-e^{-1}, 0)$, and the map is monotone in the positive combination $3.5c_1e^{-2/3} + c_2$. Directed evaluation gives the stated enclosure. The ordering of η_q and B_q^* is immediate because both denominator coefficients decrease. □

The same substitution widens the at-most-one-zero region for the product of all Dirichlet L-functions modulo q:

$$\operatorname{Re} s \geq 1 - \frac{1}{10.295195 \log q + 61.272145 V(x)}.$$

Any zero admitted there remains simple, real, and associated to a real nonprincipal character. The low-height McCurley seam has positive denominator margin $79.0723408290\ldots$.

7.2 Why the fixed summatory coefficients do not follow

The source specializes positive auxiliary functions $f_1, \ldots, f_7$. Two positive inputs to f_4, and hence to f_7, are

$$d_4(q) = 0.195\sqrt{q} + 147.735, \qquad c_{q,4} = 0.055\sqrt{q}\log q.$$

The specialization evaluates the coefficient at $q = 400001$ as though that lower endpoint supplied a uniform upper bound.

Proposition 7.2. *The displayed specialization in [2] does not establish the uniform coefficients* 5.805 *and* 7.246*. In particular:*

$$\begin{aligned}\text{the Theorem 3.4 source coefficient at } q = 10^7,\ \log x = 10^8 \\ = 11.032361704699548\ldots > 7.246,\end{aligned}$$

and, after inserting the new zero-free pair and threshold,

$$\text{the Theorem 3.3 source envelope at } q = 10^{35} = 766.0610909321805715\ldots > 5.805.$$

Proof. Substitute the stated values into the source's positive Appendix-B functions, preserving every positive term. The resulting expressions are evaluated with outward rounding. The growth floor $f_7(q) \gg q\log^2 q$ also shows structurally why a minimum at the lower endpoint cannot serve as a uniform upper bound. $\square$

Remark 7.3. *Proposition 7.2 identifies a gap in the displayed numerical deduction. It does not disprove either Goldbach summatory conclusion. Accordingly, this paper claims the valid parameter and zero-region improvements of proposition 7.1, but not new fixed-coefficient summatory theorems.*

7.3 The corrected no-exception condition

To force the possible real exceptional zero outside the widened region, one needs

$$\frac{1}{10.295195\log q + 61.272145V(x)} \le \frac{100}{\sqrt{q}\log^2 q}.$$

For $q > 400000$, define

$$A_q = \frac{\sqrt{q}\log^2 q/100 - 10.295195\log q}{61.272145}.$$

The required inequality is $V(x) \ge A_q$.

Proposition 7.4. *The sharp strict Lambert-W sufficient condition is*

$$\boxed{x > \exp\left(\exp\left(\frac{1}{2}W(2A_q^3)\right)\right).}$$

The argument $2A_q^3$ cannot be replaced by A_q^3.

Proof. Put $z = 2\log\log x$. Then

$$V(x)^3 = (\log x)^2\log\log x = \frac{1}{2}ze^z.$$

Thus $V(x)^3 > A_q^3$ is equivalent to $ze^z > 2A_q^3$, which gives the displayed strict condition on the principal branch of W. Using A_q^3 instead yields only $V(x)^3 = A_q^3/2$. $\square$

8 Certificate boundary and reproducibility

The non-computational argument consists of the reductions and monotonicity steps proved in Sections 2, 5, 6, and 7. The finite certificate has three roles:

(i) exact integer reconstruction of the degree-46 Gram polynomial and exact positivity of all cosine coefficients;

(ii) directed evaluation of smoothing roots, limiting ratios, endpoint margins, the fixed-threshold envelope, and the frontier crossing; and

(iii) directed evaluation of the Goldbach threshold and the two explicit uniformity counterchecks.

The supplementary archive contains the integer matrix, a package-native checker, a manifest of source and theorem hashes, and deterministic build instructions. The exact polynomial stage uses integer and rational arithmetic. The transcendental stage uses real-ball arithmetic at 512-bit precision and is repeated independently at higher precision. Every target inequality has a strict reserve; optimized execution that would suppress assertions is rejected.

9 Limitations

The following boundaries are part of the theorem statements.

1. The global pair applies for $|t| \geq 10$ and does not exclude a possible real exceptional zero at height zero.
2. The stronger pair $(9.93082, 48.07157)$ begins at an absolute effectively computable height Y that is not numerically evaluated.
3. No optimality is claimed for degree 46, for the fixed global threshold L_0, or among all possible zero-free-region methods.
4. The valid Goldbach propagation consists of the parameter map, B_q^* ordering, and widened at-most-one-zero region. The fixed coefficients 5.805 and 7.246 are not inherited.
5. The results do not prove the Riemann hypothesis or the generalized Riemann hypothesis.

Data and code availability

The source package includes the exact integer certificate, the arithmetic checker, source hashes, and deterministic commands required to rebuild the paper and replay the finite computations. No proprietary data are used.

Acknowledgements

The author thanks the authors of the cited explicit zero-free-region papers for making source material and numerical constants publicly available.

References

[1] C. Bellotti, *Explicit bounds for the Riemann zeta function and a new zero-free region*, J. Math. Anal. Appl. **536** (2024), article 128249.

[2] G. Bhowmik, A.-M. Ernvall-Hytönen and N. Palojärvi, *Explicit estimates for the Goldbach summatory function*, arXiv:2411.00323v2 (2025).

[3] K. Ford, *Zero-free regions for the Riemann zeta function*, in Number Theory for the Millennium, II, A K Peters, 2002, pp. 25–56.

[4] H. Kadiri, *An explicit zero-free region for the Dirichlet L-functions*, Mathematika **64** (2018), no. 2, 445–474.

[5] T. Khale, *An explicit Vinogradov–Korobov zero-free region for Dirichlet L-functions*, Q. J. Math. **75** (2024), no. 1, 299–332.

[6] K. S. McCurley, *Explicit zero-free regions for Dirichlet L-functions*, J. Number Theory **19** (1984), no. 1, 7–32.

[7] M. J. Mossinghoff, T. S. Trudgian and A. Yang, *Explicit zero-free regions for the Riemann zeta-function*, arXiv:2212.06867 (2022).

[8] J. Thorner and A. Zaman, *Refinements to the prime number theorem for arithmetic progressions*, Math. Z. **306** (2024), no. 3, article 54.

[9] A. Yang, *Explicit zero-free regions and the prime number theorem*, Ph.D. thesis, University of New South Wales, 2025.