Exact threshold cells and corrections in Axler's prime-counting tables

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Abstract

Let
$$A(x) = \log x - \frac{x}{\pi(x)}, \qquad B(x) = \log x \left(\log x - 1 - \frac{x}{\pi(x)} \right).$$
We give a complete exact audit of the coefficient/start pairs in Tables 7–9 of Axler's 2024 prime-counting estimates. Every Table 7 row is valid, has the published least integer start, and admits an exact fixed-range coefficient threshold and an exact least-start coefficient cell. In Table 8, the published coefficient 1.071 fails at the prime 22 078 033; the exact same-range threshold is 1.071000358265763676..., the coefficient 1.07100036 repairs the published range, and the original coefficient is valid from the least start 22 078 034. The remaining Table 8 rows are valid and are classified sharply. In Table 9, the coefficient 1.149 fails at its published composite endpoint 42 575 222 481; replacing the start by 42 575 222 505, or replacing the coefficient by 1.14900031, gives exact repairs. Every other Table 9 row is valid with its published least start, and all corrected starts have exact coefficient cells.

The proofs reduce the continuous inequalities to suffix maxima of A or B at the initial endpoint and prime jumps, certify the finite ranges by directed fixed-point arithmetic with exact prime-count reconciliation, and close the tails using explicit bounds of Büthe and Dusart. The largest retained finite scans have one exhaustive prime-enumeration implementation; their boundary counts, worst-point arithmetic, source rows, and analytic tails are independently reconstructed.

1 Introduction

Axler's explicit estimates include two families of upper bounds for the prime-counting function. The first has the Legendre form
$$\pi(x) < \frac{x}{\log x - c}, \tag{1}$$
and the second has the two-term form
$$\pi(x) < \frac{x}{\log x - 1 - c/\log x}. \tag{2}$$
Tables 7 and 8 of [2] list coefficient/start pairs for (1); Table 9 lists coefficient/start pairs for (2). The published proofs use direct computation for the finite ranges, but do not identify the exact suffix maxima, the complete least-start coefficient cells, or the two false rows isolated below.

The natural normalizations are
$$A(x) = \log x - \frac{x}{\pi(x)} \tag{3}$$
and
$$B(x) = \log x \left(\log x - 1 - \frac{x}{\pi(x)} \right). \tag{4}$$

When the relevant denominator is positive, (1) is equivalent to $A(x) < c$, and (2) is equivalent to $B(x) < c$. Thus validity on a half-line is a suffix-maximum problem. Minimal integer starts are governed by the value at the preceding integer, and continuous coefficient perturbations are governed by a two-ended cell.

The results have three parts.

(i) Table 7 is valid exactly as printed. Every listed start is least, is the unique suffix maximizer for A, and determines an exact coefficient cell.

(ii) Table 8 contains one false row, $c = 1.071$, and otherwise admits a complete exact classification, including the historical 1.08366 row.

(iii) Table 9 contains one false row, $c = 1.149$. Correcting its start yields a complete valid table, and every corrected start has an exact coefficient cell.

These are fixed-form, fixed-range results. They do not improve the asymptotic prime number theorem, and the coefficient cells do not constitute the complete Pareto frontier over arbitrary coefficients and starts.

2 Step-function geometry

2.1 The Legendre normalization

Lemma 2.1. *On a prime-free interval on which $\pi(x) = n$,*

$$A'(x) = \frac{1}{x} - \frac{1}{n} < 0.$$

At the next prime p_{n+1}, the jump is

$$A(p_{n+1}) - A(p_{n+1}^-) = \frac{p_{n+1}}{n(n+1)} > 0.$$

Consequently a suffix maximum of A on a finite interval beginning at an arbitrary real endpoint can occur only at that endpoint or at an actual prime.

Proof. Differentiate (3) while $\pi(x) = n$. Since $x > n$ throughout the ranges considered, the derivative is negative. The jump formula follows by replacing n by $n + 1$ at p_{n+1}. □

2.2 The two-term normalization

Lemma 2.2. *On a prime-free interval on which $\pi(x) = n$, writing $L = \log x$,*

$$B'(x) = \frac{2L - 1}{x} - \frac{L + 1}{n}.$$

For every $x \geq 9$, this derivative is negative. Hence a suffix maximum of B in the ranges considered can occur only at the initial endpoint or at an actual prime.

Proof. The derivative is direct. For $x \geq 9$, the elementary inequality $\pi(x) \leq x/2$ gives

$$B'(x) \leq \frac{2L - 1}{x} - \frac{2(L + 1)}{x} = -\frac{3}{x} < 0.$$

□

2.3 Exact least-start cells

Proposition 2.3. *Let G be either A or B, with the corresponding denominator positive on $[N-1,\infty)$. Suppose*

$$\alpha(N) = \max_{x\geq N} G(x), \qquad \beta(N) = G(N-1),$$

and G is strictly decreasing on $[N-1,N)$. Then N is the least integer start of the corresponding strict upper bound if and only if

$$\alpha(N) < c \leq \beta(N).$$

Proof. If $\alpha(N) < c$, the bound holds on $[N,\infty)$. If also $c \leq \beta(N)$, it fails at $N-1$, so no smaller integer start is valid. If $c \leq \alpha(N)$, the bound fails at a suffix optimizer. If $\beta(N) < c$, strict decrease on the predecessor unit interval and the bound from N show that $N-1$ is already valid. □

3 Tables 7 and 8

Theorem 3.1 (Complete Table 7 audit)**.** *For every coefficient/start pair (c,N) in Appendix A, the inequality*

$$\pi(x) < \frac{x}{\log x - c} \qquad (x \geq N)$$

holds, and N is the least valid integer start. Every N is composite, the unique suffix maximizer of A is N, and

$$\alpha(N) = A(N), \qquad \beta(N) = A(N-1).$$

For every real $d < \log(N-1)$, the start N is least for coefficient d if and only if

$$A(N) < d \leq A(N-1).$$

At the exact non-strict threshold $d = A(N)$, equality occurs only at $x = N$.

Theorem 3.2 (Complete Table 8 audit and correction)**.** *For every Table 8 row in Appendix B except $c = 1.071$, the published half-line is valid and the displayed start is least. The coefficient 1.071 with start $22\,078\,017$ is false at*

$$p_* = 22\,078\,033, \qquad \pi(p_*) = 1\,393\,895,$$

where

$$A(p_*) = 1.071000358265763676457088876 48\ldots.$$

The exact repairs are

$$\pi(x) < \frac{x}{\log x - 1.07100036} \qquad (x \geq 22\,078\,017)$$

and

$$\pi(x) < \frac{x}{\log x - 1.071} \qquad (x \geq 22\,078\,034),$$

with both displayed starts least.

For each literal Table 8 start N, let $\alpha(N) = \max_{x\geq N} A(x)$ and $\beta(N) = A(N-1)$. Then for $d < \log(N-1)$, N is the least start exactly when

$$\alpha(N) < d \leq \beta(N).$$

The optimizer is the displayed x_* in Appendix B. The same cell theorem holds for the corrected coefficient-1.071 start $22\,078\,034$.

Remark 3.3. The coefficient 1.08365366 in Appendix B is the strict same-range repair of the historical 1.08366 row. Its exact non-strict threshold is

$$\log(1\,526\,671) - \frac{1\,526\,671}{116\,053} = 1.0836536589641492328911007601 8\ldots.$$

4 Corrected Table 9

Theorem 4.1 (Complete corrected Table 9)**.** *Every Table 9 row in Appendix C other than the originally printed coefficient-*1.149 *start is valid on its full displayed half-line, and every displayed start is least. The published assertion*

$$\pi(x) < \frac{x}{\log x - 1 - 1.149/\log x} \qquad (x \geq 42\,575\,222\,481)$$

is false at its composite endpoint, where

$$\pi(42\,575\,222\,481) = 1\,817\,311\,115$$

and

$$B(42\,575\,222\,481) = 1.1490003091852194519030898606 0\ldots.$$

The exact practical repairs are

$$\pi(x) < \frac{x}{\log x - 1 - 1.149/\log x} \qquad (x \geq 42\,575\,222\,505)$$

and

$$\pi(x) < \frac{x}{\log x - 1 - 1.14900031/\log x} \qquad (x \geq 42\,575\,222\,481),$$

with both starts least.

Theorem 4.2 (Table 9 coefficient cells)**.** *For each corrected Table 9 start N, and for the supplementary* 1.14900031 *repair start, put*

$$\alpha(N) = \max_{x \geq N} B(x), \qquad \beta(N) = B(N-1).$$

For every

$$d < \log(N-1)(\log(N-1) - 1),$$

the integer N is the least start for coefficient d if and only if

$$\alpha(N) < d \leq \beta(N).$$

The suffix optimizer is N in every case except the coefficient 1.15 *row, for which it is*

$$p_* = 38\,284\,442\,321, \qquad \pi(p_*) = 1\,641\,621\,296.$$

5 Certified finite ranges

The finite certificates use the reductions in Lemmas 2.1 and 2.2. We summarize the directed arithmetic because it is the common core of all three table audits.

Let $S = 2^{56}$. A relative chunk begins at u, and for a prime p in that chunk put

$$z = \frac{p-u}{u}, \qquad Z = \lceil Sz \rceil, \qquad L_u = \lceil S \log u \rceil.$$

The chunks satisfy $0 \le z \le 10^{-5}$, so

$$\log(1+z) \le z - \frac{z^2}{2} + \frac{z^3}{3}.$$

Therefore

$$\Lambda = L_u + Z - \left\lfloor \frac{Z^2}{2S} \right\rfloor + \left\lceil \frac{Z^3}{3S^2} \right\rceil$$

satisfies $\log p \le \Lambda / S$. If $p = p_n$, let

$$R = \left\lfloor \frac{pS}{n} \right\rfloor.$$

For the A-tables,

$$A(p) \le \frac{\Lambda - R}{S}.$$

An exact rational coefficient comparison therefore reduces to an integer inequality after clearing denominators. For the B-table,

$$B(p) \le \frac{\Lambda(\Lambda - R - S)}{S^2};$$

again the coefficient comparison is an exact integer inequality. Candidates that are not strictly separated are rejected rather than rounded into acceptance.

The retained finite certificates cover:

scope	prime endpoints	fallback count	residual implementation caveat
Table 7	60 476 803 149	0	one exhaustive interior scan
Table 8	2 740 927 through the common cutoff	0	independent high-precision suffix reconstruction
Table 9 correction	528 206 511	0	one exhaustive interior scan
complete Table 9	491 051 669 627	0	one exhaustive segmented scan
Table 9 1.15 optimizer	175 689 820	0	one exhaustive optimizer scan

For the large scans, every segment boundary count and every reported worst-point count is independently reconstructed under two exact prime-counting algorithms. The worst-point fixed-point arithmetic is also reconstructed separately. This does not amount to a second full prime-enumeration implementation, and we retain that limitation explicitly.

6 Analytic tails

For the Legendre form define

$$F_c(x) = \frac{x}{\log x - c} - \mathrm{li}(x).$$

Then

$$(\log x)(\log x - c)^2 F_c'(x) = (c-1)\log x - c^2.$$

For the two-term form define

$$G_c(x) = \frac{x}{\log x - 1 - c/\log x} - \mathrm{li}(x).$$

Writing $L = \log x$,

$$L^3(\log x - 1 - c/\log x)^2 G_c'(x) = (c-1)L^2 - 3cL - c^2.$$

The retained interval certificates prove positivity at each selected handoff and positivity of the corresponding derivative polynomial from that point onward. Büthe's explicit result gives

$$\pi(x) < \mathrm{li}(x) \qquad (2 \le x \le 10^{19}).$$

For $x \ge 10^{19}$, Dusart's estimate gives, with $L = \log x$,

$$\pi(x) \le \frac{x}{L}\left(1 + \frac{1}{L} + \frac{2}{L^2} + \frac{7.59}{L^3}\right).$$

For the Legendre form, comparison with $x/(L-c)$ reduces to

$$(c-1)L^3 - (2-c)L^2 - (7.59 - 2c)L + 7.59c > 0.$$

For the two-term form, comparison reduces to

$$(c-1)L^3 + (c+2-7.59)L^2 + (2c+7.59)L + 7.59c > 0.$$

The coefficient derivatives of both polynomials are positive in the relevant ranges, so the smallest coefficient controls each table. Directed interval evaluations of the polynomial and its successive derivatives at $\log(10^{19})$ close every infinite tail.

7 Sharpness, crossings, and scope

At a fixed start N, the exact non-strict threshold is the suffix maximum $\alpha(N)$. Equality is attained at the unique optimizer. Strict coefficients are precisely those above $\alpha(N)$, subject to denominator positivity. Proposition 2.3 adds the upper endpoint $\beta(N)$ and gives the exact interval on which N remains the least integer start.

When N is composite, π is constant on the predecessor unit interval. The strict monotonicity there gives a unique final real equality in $(N-1, N)$. In the Legendre form it can be written

$$r = -nW_{-1}\left(-\frac{e^c}{n}\right), \qquad n = \pi(N).$$

The W_{-1} branch is forced because the relevant solution satisfies $r/n > 1$.

The conclusions are deliberately limited:

- the exact thresholds are for the displayed denominator shapes and fixed half-lines;
- the cell theorems exclude the disconnected pole regimes;
- no complete coefficient/start Pareto frontier is asserted;
- no asymptotic prime-number-theorem constant is improved;
- novelty is limited to the documented source and follow-up searches through July 19, 2026.

A Axler Table 7

Table 1: The 42 Table 7 coefficient/start pairs. Every start is least.

c	least integer start N
1.0344	98,011,218,006,714
1.0345	90,093,726,828,053
1.0346	82,972,765,680,514
1.0347	76,292,362,570,940
1.0348	70,363,470,737,452
1.0349	64,716,191,738,353
1.035	59,667,044,596,151
1.036	27,086,141,056,455
1.037	12,806,615,320,917
1.038	6,317,261,904,937
1.039	3,231,501,496,562
1.040	1,697,021,254,855
1.041	924,640,658,874
1.042	519,205,451,664
1.043	296,735,291,225
1.044	175,758,684,156
1.045	105,640,136,371
1.046	65,431,161,562
1.047	41,022,022,044
1.048	25,724,702,310
1.049	17,231,171,472
1.050	11,207,440,881
1.051	7,538,561,672
1.052	5,047,295,951
1.053	3,745,835,388
1.054	2,605,443,747
1.055	1,810,796,757
1.056	1,220,594,340
1.057	876,542,559
1.058	673,828,570
1.059	501,155,566
1.060	383,446,375

c	least integer start N
1.061	269,585,283
1.062	196,894,353
1.063	180,220,137
1.064	116,749,925
1.065	110,166,540
1.066	76,223,058
1.067	53,431,171
1.068	46,097,944
1.069	39,706,453
1.070	31,027,247

B Axler Table 8

Table 2: Table 8 starts, exact suffix optimizers, and strict repairs. The 1.071 row is repaired rather than asserted as printed.

published c	start N	optimizer x_*	strict repair
1.071	22,078,017	22,078,033	1.07100036
1.072	18,339,738	18,339,738	1.071999606
1.073	13,026,859	13,026,859	1.072999464
1.074	12,895,928	12,895,928	1.073999227
1.075	8,832,927	8,832,927	1.074999064
1.076	7,299,254	7,299,254	1.075999344
1.077	7,117,256	7,117,303	1.076998803
1.078	5,465,656	5,465,671	1.077999313
1.079	4,994,010	4,994,010	1.078999622
1.080	3,462,478	3,462,478	1.079996469
1.081	3,455,648	3,455,648	1.080997538
1.082	2,279,177	2,279,177	1.081994826
1.083	1,529,630	1,529,630	1.082994823
1.08366	1,526,671	1,526,671	1.08365366
1.084	1,525,432	1,525,432	1.083997108
1.085	1,515,074	1,515,074	1.084996533
1.086	1,200,014	1,200,014	1.085998378
1.087	1,195,296	1,195,296	1.086994716
1.088	624,878	624,878	1.087995115
1.089	618,726	618,726	1.088995383
1.090	618,058	618,058	1.089988672
1.091	445,112	445,112	1.090991674
1.092	359,804	359,804	1.091991098
1.093	356,203	356,203	1.092970546
1.094	355,990	355,990	1.093992304
1.095	355,177	355,177	1.094995652

published c	start N	optimizer x_*	strict repair
1.096	155,935	155,935	1.095942182
1.097	155,907	155,907	1.096956443
1.098	60,297	60,297	1.097916842
1.099	60,224	60,224	1.098933642

C Corrected Axler Table 9

Table 3: The corrected complete Table 9. Only the printed start for $c = 1.149$ changes.

c	least integer start N
1.11	62,998,850,942,976
1.1105	55,193,608,062,217
1.111	49,246,036,992,716
1.112	38,472,138,880,411
1.113	30,658,643,813,468
1.114	23,767,640,743,883
1.115	19,278,513,358,342
1.116	15,142,627,022,527
1.117	12,279,648,138,508
1.118	9,684,114,630,824
1.119	7,981,446,192,206
1.12	6,323,967,140,812
1.121	5,273,225,700,761
1.122	4,170,462,893,841
1.123	3,458,549,136,539
1.124	2,825,539,807,244
1.125	2,292,448,124,593
1.126	1,903,596,231,542
1.127	1,573,767,234,188
1.128	1,290,096,268,844
1.129	1,073,403,839,693
1.13	889,377,392,161
1.131	782,989,678,664
1.132	608,408,258,090
1.133	540,050,850,157
1.134	452,875,824,702
1.135	373,479,021,700
1.136	335,562,521,091
1.137	263,728,502,964
1.138	242,118,904,367
1.139	201,924,836,111
1.14	161,054,192,492
1.141	149,061,190,565

c	least integer start N
1.142	125,233,112,846
1.143	105,053,836,224
1.144	86,061,321,374
1.145	77,278,924,451
1.146	61,344,524,412
1.147	57,720,831,343
1.148	46,039,922,948
1.149	42,575,222,505
1.15	38,284,442,297

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