Primes between consecutive sixty-fourth powers and optimal Perron weights

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July 22, 2026

Abstract

We prove that every interval between consecutive perfect sixty-fourth powers contains a prime. More precisely, for every integer $n \geq 1$ there is a prime p with

$$n^{64} < p < (n+1)^{64}.$$

The proof combines an explicit Riemann–von Mangoldt formula, a full-strip explicit zero-density estimate, a classical and a Littlewood zero-free region, and the verified critical-line location of the zeta zeros through height 3000175332800. A directed interval computation checks 64,432 closed logarithmic cells, with worst normalized error $0.9987282794\ldots$, and an analytic argument treats the infinite tail. We also solve two variational problems arising in the averaged Perron formula. On the fixed range $[T, 2T]$, the unique nonnegative normalized C^1 weight minimizing the source norm is the constant weight, improving the Perron coefficient from 1.785 to $(3+2\log 2)/\pi$. For an arbitrary averaging ratio we identify the exact phase transition from constant weights to right-tail steps. Finally, a weighted coarea formula reduces every linear norm–moment optimization to a compact two-endpoint problem. These optimization results clarify the method but do not assert that exponent 64 is optimal.

Keywords. primes in short intervals; consecutive perfect powers; explicit formula; zero-density estimate; averaged Perron formula; interval arithmetic; weighted coarea.

2020 Mathematics Subject Classification. Primary 11N05; Secondary 11M06, 11M26, 26A45, 65G20.

1 Introduction

The problem of locating primes between consecutive perfect powers is a classical explicit form of the prime-in-short-interval problem. Ingham's zero-density method gives primes in intervals of length $x^{5/8+\varepsilon}$ for sufficiently large x, and later asymptotic work has reached considerably shorter intervals; see, for example, [5, 1]. Such asymptotic results do not by themselves cover every positive integer. A theorem valid between n^k and $(n+1)^k$ for all n requires explicit constants, a finite-range splice, and complete control of every parameter seam.

Cully-Hugill and Johnston developed explicit truncated Riemann–von Mangoldt formulae suited to this problem [3, 4]. Lee subsequently combined their method with explicit zero-free regions and proved that every interval between consecutive perfect 86th powers contains a prime [6]. Lee also isolated the precise zero-free regions that would suffice for smaller exponents. The present paper changes two parts of that calculation. First, it uses a full-strip explicit Huxley–Ingham density estimate with height-dependent constants. Second, it retunes the arbitrary-parameter Riemann–von Mangoldt remainder and verifies the resulting inequality on closed intervals.

Our main theorem is the following.

Theorem 1.1. *For every integer $n \geq 1$, there exists a prime p satisfying*

$$n^{64} < p < (n+1)^{64}.$$

The proof first establishes the stronger explicit statement

$$\theta(x + 64x^{63/64}) - \theta(x) > 0 \qquad (x \geq e^{1946}), \tag{1}$$

where

$$\theta(x) = \sum_{p \leq x} \log p.$$

Lee's finite-range theorem overlaps (1), and the binomial theorem then gives the strict upper endpoint

$$n^{64} + 64n^{63} < (n+1)^{64}.$$

The numerical certificate is not a point scan. Its finite part covers

$$1946 \leq L = \log x \leq 10000$$

by 64,432 contiguous closed intervals of width 1/8. Each cell contains one exact rational height choice. Directed Arb arithmetic checks the source domains, zero-free branch, density band, positive integration slopes, and the strict inequality $\mathcal{E}(L) < 1$ on the entire cell. The worst independent value is

$$\mathcal{E}(3512.375) = 0.998728279409124286\ldots.$$

The range $L \geq 10000$ is handled by a decreasing analytic majorant below 1/2.

The same Perron framework contains a useful variational problem. For a nonnegative normalized C^1 weight on $[1, \xi]$, define

$$J_\xi(w) = w(1) + \frac{w(\xi)}{\xi} + \int_1^\xi \frac{w(u)}{u}\, \mathrm{d}u + \int_1^\xi \frac{|w'(u)|}{u}\, \mathrm{d}u. \tag{2}$$

This is 2π times the $k = 1$ source norm arising from the weighted Perron method of Ramaré and Cully-Hugill–Johnston [8, 4]. We prove three exact results about (2).

Theorem 1.2. *Among all nonnegative normalized C^1 weights on $[1, 2]$,*

$$J_2(w) \geq \frac{3}{2} + \log 2,$$

with equality only for $w(u) = 1$. Consequently the optimal fixed-range Perron coefficient in this class is

$$C_{\mathrm{P}} = \frac{3 + 2\log 2}{\pi} = 1.3962008588566752014\ldots < 1.397.$$

For real-valued Perron remainders, the pointwise T^*-corollary of [4] remains valid with C_{P} in place of 1.785.

Theorem 1.3. *There is a unique number*

$$\xi_c = 2.110355059880694877\ldots$$

satisfying

$$3 + \xi_c^{-1} + \log \xi_c - 2\xi_c = 0.$$

If $1 < \xi \le \xi_c$, *then*

$$\inf J_\xi(w) = \frac{1 + \xi^{-1} + \log \xi}{\xi - 1},$$

and the unique minimizer is the constant weight $(\xi - 1)^{-1}$. *If* $\xi > \xi_c$, *let* $a \in (1, \xi)$ *be the unique root of*

$$1 + \frac{2}{a} + \log \frac{\xi}{a} - \frac{(a+1)\xi}{a^2} + \frac{1}{\xi} = 0.$$

Then

$$\inf J_\xi(w) = \frac{a+1}{a^2}.$$

The latter infimum is not attained in C^1, *but is approached by smooth right-tail steps supported on* $[a, \xi]$.

Theorem 1.4. *Let*

$$\mu_\xi(w) = \int_1^\xi u w(u) \, \mathrm{d}u.$$

For every nonnegative normalized C^1 *weight* w, *there is a probability measure* ν_w *on* $\{(a,b) : 1 \le a < b \le \xi\}$ *such that*

$$J_\xi(w) = \int \frac{a^{-1} + b^{-1} + \log(b/a)}{b - a} \, \mathrm{d}\nu_w(a,b),$$

$$\mu_\xi(w) = \int \frac{a+b}{2} \, \mathrm{d}\nu_w(a,b).$$

Consequently, for every $\beta \in \mathbb{R}$,

$$\inf_w \{J_\xi(w) + \beta \mu_\xi(w)\} = \min_{1 \le a < b \le \xi} \frac{a^{-1} + b^{-1} + \log(b/a) + \frac{\beta}{2}(b^2 - a^2)}{b - a}.$$

The optimization theorems do not imply that the complete nonlinear explicit-formula remainder is minimized by one interval step. We make the obstruction precise in section 7: the relevant cutoff function has indefinite Hessian. In particular, none of these results claims that exponent 64 is the least accessible exponent, or that the corresponding statement for 63rd powers is false.

2 Explicit inputs and notation

Write

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, \ \beta \ge \sigma, \ 0 < \gamma \le T\},$$

with multiplicity. We use four explicit inputs.

2.1 Verified low zeros

Platt and Trudgian proved that every nontrivial zero through

$$H_R = 3000175332800 \tag{3}$$

lies on the critical line [7]. We use only critical-line location, not simplicity.

2.2 A full-strip density estimate

Chourasiya and Simonič gave an explicit form of the Huxley–Ingham estimate on the sixteen intervals $[j/32, (j+1)/32]$, $16 \le j \le 31$ [2]. Reoptimizing the coupled σ-dependent factors gives the following form needed below.

Proposition 2.1. *For $T \ge 3 \cdot 10^{12}$, the lower fifteen strips satisfy*

$$N(\sigma, T) \le 22.062 T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^{\frac{7-5\sigma}{2-\sigma}} + 9.461 \log^2 T + 167.8 \log T$$

for $1/2 \le \sigma \le 31/32$.

For the top strip, put $\delta = 1 - \sigma$, $\delta_0 = 1/32$, and

$$A(\delta) = \frac{3\delta}{1+\delta}, \quad P_1(\delta) = \frac{2+5\delta}{1+\delta}, \quad P_2(\delta) = \frac{2-\delta}{1+\delta},$$

$$Q(\delta) = \frac{1}{1+\delta} - \frac{1}{1+\delta_0}.$$

On each height band used by the certificate,

$$\begin{aligned} N(1-\delta, T) \le{}& M_1 T^{A(\delta)} (\log T)^{P_1(\delta)} e^{-3Q(\delta)/2} \\ &+ M_2 T^{A(\delta)} (\log T)^{P_2(\delta)} e^{-9Q(\delta)/5} \\ &+ B_2 \log^2 T + B_3 \log T, \end{aligned}$$

where the active coefficients are listed in table 1.

The source's three coupled factors have logarithmic derivative

$$\frac{2 \log L_0 + 3 \log(K_3/K_j)}{(2-\sigma)^2}.$$

This determines the correct common endpoint rather than multiplying three separately optimized maxima. The directed calculation also retains the source's lower-order terms. The narrow bands near $\log T = 56.6$ are necessary because the final normalized error has little spare room.

2.3 Zero-free regions

We use

$$\nu(u) = \max\left\{ \frac{1}{4.8594u}, \frac{\log u}{19.62u} \right\}. \tag{4}$$

The Littlewood component is Yang's theorem [9]. The classical component is a directed sharpening of Yang's denominator 4.862.

Table 1: Directed top-strip coefficient caps.

$\log T_0$	M_1	M_2	B_2	B_3
30	14.46	10.70	4.802	85.15
34	13.41	10.20	4.733	83.94
38	12.61	9.80	4.680	82.99
42	11.96	9.47	4.636	82.21
46	11.41	9.27	4.599	81.55
50	11.14	9.20	4.568	81.00
54	10.90	9.14	4.541	80.53
56	10.80	9.12	4.529	80.32
56.45	10.7731	9.1048	4.5265	80.2681
56.50	10.7706	9.1042	4.5262	80.2631
56.55	10.7681	9.1036	4.5260	80.2581
56.60	10.7656	9.1030	4.5257	80.2531
56.65	10.7632	9.1024	4.5254	80.2481
56.70	10.7607	9.1018	4.5251	80.2432
56.75	10.7583	9.1011	4.5248	80.2382
60	10.61	9.07	4.508	79.94

Proposition 2.2. *For every $t \geq 2$,*

$$\zeta(\sigma + it) \neq 0 \qquad \textit{if} \qquad \sigma > 1 - \frac{1}{4.8594 \log t}.$$

Proof boundary. The finite-height detector is the zero-sum inequality

$$D\eta \log t > C_1(\mu) + C_2(\eta) - 10^{-7},$$

where

$$D = 5.03905010404519\ldots,$$
$$C_1(\mu) = 0.87637 + 0.12002\mu + 0.01017\mu^2 - 0.00073\mu^3,$$
$$C_2(\eta) = 13.47\eta - 167\eta^2 - 8300\eta^3.$$

Starting from Yang's amplitude $1/4.862$, one advances the amplitude in steps of width at most 10^{-100}. Uniformly for $\log H_R \leq L \leq 56.693$, the auxiliary variables satisfy

$$\mu > 1 - \frac{2.78}{L}, \qquad \eta > \frac{1}{6L}.$$

Substitution reduces the contradiction to a rational polynomial $B(L)$, decreasing on this range, with

$$\frac{B(56.693)}{D} - \frac{1}{4.8594} > 9.7893 \cdot 10^{-6}.$$

After each local step, Yang's Littlewood region restores the global hypothesis because

$$\exp\left(\frac{19.62}{4.8594}\right) = 56.6864666\ldots < 56.693.$$

The exact quotient and final remainder of the 10^{-100}-step iteration are checked with integer arithmetic. Below H_R, (3) places every zero on the critical line. These directed checks are included in the verification archive. ∎

2.4 The low-range theorem

Lee's explicit finite computation implies, for $k = 64$,

$$\theta(x + 64x^{63/64}) - \theta(x) > 0 \tag{5}$$

through the range

$$4 \cdot 10^{18} \le x \le e^{L_0},$$

where

$$\begin{aligned} L_0 &= 64 \log\left(64 \cdot 2.51949 \cdot 10^{11} \cdot \left(1 - \frac{1}{2.51949 \cdot 10^{11}}\right)^2\right) \\ &= 1946.328038808025526\ldots. \end{aligned} \tag{6}$$

3 The corrected explicit-formula remainder

Set

$$h = 64x^{63/64}, \qquad L = \log x, \qquad s = \frac{h}{x} = 64e^{-L/64},$$
$$X = x + h = x(1+s), \qquad Z = \log X = L + \eta, \qquad \eta = \log(1+s),$$

and choose

$$T = \frac{e^u}{2}, \qquad U = 2T = e^u.$$

The exact source parameters are

$$\alpha = \frac{817}{50000}, \qquad \omega = \frac{397}{500}, \qquad D = \frac{51}{100}, \qquad \lambda = \frac{1177}{20000}. \tag{7}$$

3.1 The contour constant

The Appendix-B envelope of [3] is reconstructed from its complete formula rather than from rounded tables. In particular, the positive Z_2 term

$$4\left(\frac{2}{3}e^{3/4} + e^{1/2}\right)$$

is retained inside the source's outer factor 2, and the term printed in the source as $\nu\pi^2/24$ is used literally. Directed 384-bit arithmetic proves

$$K < 1.421 \tag{8}$$

on every finite zero-free branch and on an analytic tail. The independent maximum is near

$$L = 2790.352971090578\ldots, \qquad K = 1.420329284309273\ldots.$$

Substitution of (8) and (7) into the Section 4 formula of [4] gives

$$M = 3.541574471107266\ldots < 3.542.$$

The source prints $1/(1-\kappa) = \log x$ with $\kappa = 1 + 1/\log x$. We use the intended positive identity

$$\frac{1}{\kappa - 1} = \log x.$$

3.2 One common contour

The interval Perron theorem requires one common $T^* \in [T, 2T]$. The $X = x + h$ integral is naturally written on

$$\kappa_X = 1 + \frac{1}{Z},$$

whereas the common interval line is

$$\kappa_x = 1 + \frac{1}{L}.$$

Since $Z > L$, the rectangle between these lines lies in $\Re z > 1$, and no pole or zero is crossed. The horizontal segments contribute at most

$$\Delta K \le \frac{e^{1+\eta/L}\log(1+\eta/L)}{\pi Z^{1-\omega}} < 3.727 \cdot 10^{-16}.$$

The common-line Perron remainders must also be compared. With $a = \eta/L$, their five ratios to the base-x majorants are

$$\frac{e^a}{(1+a)^{3-\omega}}, \qquad R_{\mathcal{E}_1}, \qquad e^{-\eta/2}(1+a)^{\omega-1},$$

$$e^{-2\eta/3}(1+a)^{\omega-1}, \qquad e^{-\eta}(1+a)^{\omega}.$$

The local Brun–Titchmarsh factor defining $R_{\mathcal{E}_1}$ is decreasing in its spatial argument. The other four ratios are below one by $\omega < 1$, $a < 2 - \omega$, and $\omega \log(1+a) < \eta$. The directed check also verifies the source conditions

$$T > Z^2, \qquad T < \frac{x^\alpha - 2}{4}, \qquad \lambda \ge \frac{\theta'}{T}.$$

We may therefore reserve

$$M_{\text{int}} = 3.543. \tag{9}$$

4 The normalized error

Lee's explicit-formula reduction, with the full-strip density estimate inserted at split point $\sigma_1 = 1/2$, gives

$$\theta(x+h) - \theta(x) > 0$$

whenever the normalized error

$$\mathcal{E} = E_0 + E_{\text{dens}}^{(16)} + E_{\text{dens}}^{(<16)} + E_{\text{trunc}} + E_2 + E_3$$

is below one.

The elementary terms are

$$E_0 = \frac{u}{\pi} e^{-L/2+u},$$

$$E_{\text{trunc}} = \frac{2M_{\text{int}}}{64} e^{-(u-L/64)} \left[(1+s)Z^{1-\omega} + L^{1-\omega}\right],$$

$$E_2 = \frac{(1 + 1.93378 \cdot 10^{-8})\sqrt{1+s} - 0.999}{64} e^{(1/64-1/2)L},$$

$$E_3 = \frac{(1+1.936\cdot 10^{-8})(1+s)^{1/3} - 0.885}{64} e^{(1/64-2/3)L}.$$

For the top density strip, evaluate $\nu = \nu(u)$ from (4) and define

$$\Phi_1 = \log M_1 + uA(\nu) + P_1(\nu)\log u - L\nu - \frac{3}{2}Q(\nu),$$
$$\Phi_2 = \log M_2 + uA(\nu) + P_2(\nu)\log u - L\nu - \frac{9}{5}Q(\nu),$$

$$m_1 = L - \frac{3u + 3\log u + 3/2}{(1+\nu)^2},$$
$$m_2 = L - \frac{3u - 3\log u + 9/5}{(1+\nu)^2}.$$

Both slopes are positive. Integrating the exponential majorants from their left endpoints gives

$$E_{\mathrm{dens}}^{(16)} \le 2L\left(\frac{e^{\Phi_1}}{m_1} + \frac{e^{\Phi_2}}{m_2}\right) + 2(B_2u^2 + B_3u)\left(e^{-L\nu} - e^{-L/32}\right). \tag{10}$$

The lower fifteen strips contribute at most

$$\frac{2L\cdot 22.062 e^{u/11+(23/11)\log u - L/32}}{L - \dfrac{3u+3\log u}{(33/32)^2}} + 2(9.461u^2 + 167.8u)\left(e^{-L/32} - e^{-L/2}\right). \tag{11}$$

Proposition 4.1. *For every $L \in [1946, 10000]$, there is an exact rational choice of u for which all source conditions hold and*

$$\mathcal{E}(L) < 1.$$

Proof. Partition the interval into the closed cells

$$\left[\frac{j}{8}, \frac{j+1}{8}\right], \qquad 1946 \le \frac{j}{8} < 10000.$$

There are 64,432 cells. A schedule file stores one $u \in 10^{-6}\mathbb{Z}$ for each cell. On the full interval ball, the verifier checks:

(i) the two logarithmic source-window inequalities;

(ii) the Lee parameter window and $\nu(u) < 1/32$;

(iii) every possible active zero-free branch when interval comparison is ambiguous;

(iv) the active height-dependent density tuple and all positive integration slopes; and

(v) the sum of (10), (11), and the elementary terms strictly below one.

The worst closed cell is

$$[3512.25, 3512.375],$$

using $u = 56.65$. An independent 80-digit implementation evaluates the endpoint as

$$\mathcal{E}(3512.375) = 0.998728279409124286\ldots < 1.$$

□

Proposition 4.2. *For $L \geq 10000$, the choice*

$$u = \frac{L}{64} + 4 + \log\left(\frac{L}{10000}\right)$$

satisfies all source conditions and

$$\mathcal{E}(L) < 0.410897487735205 < \frac{1}{2}.$$

Proof. One has

$$\frac{u}{L} \leq \frac{1}{62}, \qquad L\nu \geq \frac{3100}{981}\log u.$$

Using $m_j \geq 9L/10$, the final density band, and $L/64 \leq u \leq L/62$, the two top-strip main terms are at most

$$\frac{20}{9}\left[10.61\left(\frac{L}{64}\right)^{\frac{23}{11}+\frac{50}{327}-\frac{3100}{981}} + 9.07\left(\frac{L}{64}\right)^{2+\frac{50}{327}-\frac{3100}{981}}\right],$$

and the lower-order top-strip terms are at most

$$2\left[4.508\left(\frac{L}{64}\right)^{2-\frac{3100}{981}} + 79.94\left(\frac{L}{64}\right)^{1-\frac{3100}{981}}\right].$$

The lower strips are bounded by

$$\frac{20}{9}\,22.062\left(\frac{L}{62}\right)^{23/11} e^{-(1/32-1/(62\cdot 11))L}$$

and

$$2\left[9.461\left(\frac{L}{62}\right)^{2} + 167.8\left(\frac{L}{62}\right)\right]e^{-L/32}.$$

All four expressions decrease from $L = 10000$. The elementary terms also decrease there; in particular the Perron term is bounded by a constant multiple of $L^{-\omega}$. Directed evaluation at the left endpoint gives the displayed bound. □

Proof of theorem 1.1. Propositions 4.1 and 4.2 prove (1). The low-range theorem (5) reaches beyond the analytic start because $L_0 > 1946$. Also

$$2^{64} > 4 \cdot 10^{18}.$$

Thus, for every $n \geq 2$, the point $x = n^{64}$ lies in the union of the low and analytic ranges, and there is a prime in

$$(n^{64}, n^{64} + 64n^{63}].$$

The omitted terms in the binomial expansion are positive, so

$$n^{64} + 64n^{63} < (n+1)^{64}.$$

For $n = 1$, use the prime 2. □

5 The optimal fixed-range weight

For a nonnegative normalized C^1 weight w on $[1,2]$,

$$J_2(w) = w(1) + \frac{w(2)}{2} + \int_1^2 \frac{w(u)}{u}\,\mathrm{d}u + \int_1^2 \frac{|w'(u)|}{u}\,\mathrm{d}u.$$

Set

$$A = \frac{3}{2} + \log 2, \qquad \phi(u) = 1 + \log u - A(u-1).$$

Then $\phi(1) = 1$, $\phi(2) = -1/2$, and

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u} - \phi(u)\right) = A - \frac{1}{u} - \frac{1}{u^2} \geq A - 2 > 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\phi(u) + \frac{1}{u}\right) = \frac{u-1}{u^2} - A \leq \frac{1}{4} - A < 0.$$

Hence $|\phi(u)| \leq 1/u$, strictly in the interior. Integration by parts gives

$$\begin{aligned} J_2(w) &\geq w(1) + \frac{w(2)}{2} + \int_1^2 \frac{w(u)}{u}\,\mathrm{d}u + \int_1^2 \phi(u)w'(u)\,\mathrm{d}u \\ &= A\int_1^2 w(u)\,\mathrm{d}u = A. \end{aligned}$$

Equality forces $w' = 0$, proving theorem 1.2.

For the constant weight,

$$\mathcal{L}_2 = \frac{1}{\pi}\int_1^2 u\,\mathrm{d}u = \frac{3}{2\pi}.$$

The induced cutoff is the unique positive root of

$$3\vartheta^3 + 2\pi\vartheta^2 - (3 + 2\log 2) = 0,$$

and directed evaluation gives

$$0.72068 < \vartheta < 0.72069.$$

The weighted averaged theorem of [4] remains valid for arbitrary complex coefficients. The pointwise inference requires a real-valued remainder. Indeed, if a continuous real function G satisfied $|G(t)| > R$ throughout the connected interval $[T, 2T]$, then G would have constant sign, contradicting a nonnegative weighted-average bound by R. The analogous complex statement is false: $G(t) = e^{2\pi i(t-T)/T}$ has average zero and modulus one everywhere.

Substituting C_{P}, the safe cutoff floor 0.72068, and $\lambda = 71/1250$ into the parameters of section 3 gives

$$M < 3.262, \qquad M_{\text{int}} = 3.263.$$

This strengthens every inequality used for theorem 1.1. The original certificate retains the larger reserve (9), so the main proof does not depend on a numerical recertification after the variational improvement.

6 The all-ratio phase transition

For $1 < \xi \le \xi_c$, put

$$A(\xi) = \frac{1 + \xi^{-1} + \log \xi}{\xi - 1}$$

and

$$\phi(u) = 1 + \log u - A(\xi)(u - 1).$$

The identity

$$(A(\xi) - 2)(\xi - 1) = 3 + \xi^{-1} + \log \xi - 2\xi$$

shows that $A(\xi) \ge 2$ exactly on this range. As in section 5,

$$|\phi(u)| \le \frac{1}{u}, \qquad \phi(1) = 1, \qquad \phi(\xi) = -\frac{1}{\xi}.$$

Integration by parts yields $J_\xi(w) \ge A(\xi)$, with equality only for the normalized constant weight.

Now suppose $\xi > \xi_c$, and define

$$F_\xi(a) = 1 + \frac{2}{a} + \log \frac{\xi}{a} - \frac{(a+1)\xi}{a^2} + \frac{1}{\xi}.$$

Then

$$F_\xi(1) < 0, \qquad F_\xi(\xi) = \frac{2}{\xi} > 0,$$

$$F_\xi'(a) = \frac{(\xi - a)(a + 2)}{a^3} > 0.$$

Thus F_ξ has one root $a \in (1, \xi)$. Set

$$c = \frac{a+1}{a^2}$$

and use the piecewise C^1 dual function

$$\phi(u) = \begin{cases} u^{-1}, & 1 \le u \le a, \\ a^{-1} + \log(u/a) - c(u - a), & a \le u \le \xi. \end{cases}$$

The root equation is exactly $\phi(\xi) = -1/\xi$. On $[a, \xi]$,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u} - \phi(u)\right) = c - \frac{1}{u} - \frac{1}{u^2} \ge 0,$$

while $\phi(u) + 1/u$ decreases to zero. Hence $|\phi(u)| \le 1/u$. The dual density is

$$\frac{1}{u} - \phi'(u) = \begin{cases} u^{-1} + u^{-2}, & 1 \le u \le a, \\ c, & a \le u \le \xi, \end{cases}$$

so $J_\xi(w) \ge c$.

In the bounded-variation relaxation, the normalized right-tail step

$$w_{\mathrm{BV}}(u) = \frac{\mathbf{1}_{[a,\xi]}(u)}{\xi - a}$$

has

$$J_\xi(w_{\mathrm{BV}}) = \frac{\xi^{-1} + \log(\xi/a) + a^{-1}}{\xi - a} = c.$$

The last equality is $F_\xi(a) = 0$. Replacing the jump by the cubic smoothstep $3t^2 - 2t^3$ and renormalizing gives nonnegative normalized C^1 weights converging to this value. Equality cannot hold in C^1: it would force $w = 0$ on $[1, a)$, $w' = 0$ on (a, ξ), and then $w = 0$ everywhere by continuity. This proves theorem 1.3.

7 Weighted coarea and the Pareto frontier

Let $E_t = \{u : w(u) > t\}$. For almost every $t > 0$, write the components of E_t as $(a_{t,j}, b_{t,j})$. The weighted one-dimensional coarea formula, together with the endpoint charges in (2), gives

$$w(1) + \frac{w(\xi)}{\xi} + \int_1^\xi \frac{|w'(u)|}{u}\,\mathrm{d}u = \int_0^\infty \sum_j \left(\frac{1}{a_{t,j}} + \frac{1}{b_{t,j}}\right)\mathrm{d}t.$$

An endpoint touching 1 or ξ contributes the corresponding trace charge; an interior endpoint contributes weighted variation. Layer cake gives

$$\int_1^\xi \frac{w(u)}{u}\,\mathrm{d}u = \int_0^\infty \sum_j \log\frac{b_{t,j}}{a_{t,j}}\,\mathrm{d}t,$$

$$\mu_\xi(w) = \int_0^\infty \sum_j \frac{b_{t,j}^2 - a_{t,j}^2}{2}\,\mathrm{d}t,$$

and normalization gives

$$1 = \int_0^\infty \sum_j (b_{t,j} - a_{t,j})\,\mathrm{d}t.$$

Assign probability mass

$$(b_{t,j} - a_{t,j})\,\mathrm{d}t$$

to each component interval. This proves the barycentric identities in theorem 1.4. Taking the barycenter of

$$\frac{a^{-1} + b^{-1} + \log(b/a)}{b - a} + \beta\frac{a + b}{2}$$

gives the lower bound in the support formula. Conversely, the normalized interval step

$$\frac{\mathbf{1}_{[a,b]}}{b - a}$$

has exactly the displayed norm and moment in the bounded-variation relaxation, and cubic smoothing of its jumps proves sharpness in the C^1 class.

Remark 7.1. *The coarea theorem is linear. Let $\vartheta > 0$ be defined implicitly by*

$$J = \pi\vartheta^2 + \mu\vartheta^3.$$

For $B, D > 0$, set

$$G(J, \mu) = B\frac{J}{\vartheta} + D\frac{J}{\vartheta^2}.$$

Direct implicit differentiation gives

$$\det \nabla^2 G = -\frac{\pi^2(B\vartheta + 2D)^2}{\vartheta^2(3\mu\vartheta + 2\pi)^4} < 0.$$

Thus G is a saddle. Jensen's inequality cannot reduce the full nonlinear cutoff optimization to one interval step.

8 Verification boundary and limitations

The analytic proof is separated from finite computation as follows. The source identities, common-contour comparisons, density formulas, tail monotonicity, low-range overlap, dual functions, coarea identities, and smoothing limits are stated and proved in the text. The finite certificate performs directed interval evaluation of the remaining explicit inequalities.

The primary exponent-64 replay uses Julia 1.12.6, Nemo 0.56.1, and 384-bit Arb arithmetic. It reconstructs the contour constant, the eight new density bands, all 64,432 finite cells, and the analytic tail. An independent Python implementation uses 80 decimal digits and separate formula code. The variational certificates use exact SymPy identities and 256-bit python-flint Arb arithmetic. Every source archive, schedule, proof script, and dependency is pinned by SHA-256 in the retained verification record.

The result is unconditional but finite-architecture. It does not prove that 64 is the least exponent for which the statement is true, the least exponent accessible to explicit-formula methods, or an optimal parameter choice inside the present density calculation. It does not rule out exponent 63, a different averaging interval, higher-order Perron weights, sign-changing weights with another pointwise extraction theorem, or a different explicit-formula architecture.

Data and code availability

The public manuscript source accompanies this preprint. The complete verification package, including the exact schedule, pinned source inputs, programs, and deterministic replay instructions, is retained for independent audit. No experimental or personal data were used.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of competing interest

The author declares no competing interests.

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