CERTIFIED DETECTION AND REMOVAL OF MIXED CONFLUENT FROISSART SHELLS FROM FINITE NOISY PADÉ DATA

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Abstract. Froissart doublets are usually detected by pairing a pole with a nearby zero or by thresholding a simple residue. Those tests do not classify repeated factors, mixed shells, coincident true and spurious multiplicities, or common cancellation across several Hermite–Padé channels. We give a finite-data theory that does. In exact arithmetic, a block-Hankel shift pencil recovers the cyclic minimal realization of a scalar Padé or common-denominator Hermite–Padé coefficient sequence. Its characteristic polynomial is the minimal common denominator, while the omitted nominal factors are exactly the common polynomial divisor detected by all-channel confluent numerator jets. Under coefficient errors with known bounds, including analytic Taylor tails, an explicit Hankel gap and pivot condition bound the recovered realization and denominator. Resolvent-separated contours preserve total algebraic multiplicity without pretending that a repeated-root partition is stable. Comparing these counts with certified nominal-denominator counts, then imposing two disjoint Hermite-jet margins, yields a removable/genuine/unresolved rule with zero false removals on a stated observability class. We prove compact-set error bounds for both local common factor division and root-pairing-free minimal-model reconstruction. Finally, intersecting error balls and perturbations of Jordan blocks show that the observability floor and contour-level multiplicity are unavoidable.

1. Introduction

A Padé approximant can contain poles that do not represent singularities of the function being approximated. In the familiar simple case, such a pole is accompanied by a nearby zero or a small partial-fraction residue and is called a Froissart doublet. In finite precision, however, the object to be classified need not be a single simple pair. A nominal denominator may have several excess factors of equal modulus, nonreal factors mixed with real ones, repeated roots, or extra multiplicity at the location of a genuine

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pole. For common-denominator Hermite–Padé approximation, cancellation must also occur in every numerator channel.

Classical Padé theory supplies exact denominator recurrences [1]. Robust Padé algorithms based on singular-value truncation suppress many numerical doublets [7], but a well-conditioned defining Toeplitz matrix alone does not exclude spurious poles [8]. Quantitative coprimeness and simple-doublet results therefore involve Sylvester-type matrices and spherical-derivative control [3, 4]. Matrix-pencil methods give effective pole estimators [10, 6], while confluent Prony analysis describes the severe dependence on node geometry and highest residue jets [2]. Approximate polynomial greatest common divisors [9] and asymptotic system-pole results for Hermite–Padé rows [5] address other parts of the same problem.

The purpose of this paper is to put the finite pieces into one theorem with visible stopping conditions. The main conclusions are as follows.

(1) In exact arithmetic, the block-Hankel rank is the degree of the minimal common denominator. Every invertible matching pivot of the Hankel pair gives a matrix similar to the cyclic minimal realization. It therefore recovers all reciprocal poles and exact confluent multiplicities.
(2) A nominal denominator factor is removable exactly when the corresponding all-channel Hermite jets vanish. The same confluent Vandermonde map measures the conditioning of approximate divisibility and principal-part residues.
(3) Under bounded coefficient perturbations, a separated Hankel class and an observed nonsingular pivot give explicit bounds for the realization, denominator coefficients, and Riesz projectors. Finite contour sampling produces a continuous resolvent certificate.
(4) Nominal and minimal algebraic counts yield a stable excess multiplicity. A separate all-channel jet margin is required before a particular local factor is removed. This distinction prevents unstable root pairing from entering the theorem.
(5) Both Euclidean shell division and minimal-model reconstruction have explicit uniform error bounds away from retained poles.

The separated-class qualification in item 3 is essential. Finite data with bounded error cannot exclude an additional pole whose sampled amplitude lies inside the error radius. Likewise, the Jordan partition of a repeated pole is discontinuous. Thus the stable output is total algebraic multiplicity inside a separated contour, not a list of perturbatively meaningful rootwise multiplicities.

2. Exact finite rational data

Let

$$\mathbf{R}(z) = \frac{\mathbf{P}(z)}{Q(z)} = \left(\frac{P_1(z)}{Q(z)}, \dots, \frac{P_d(z)}{Q(z)}\right), \qquad Q(0) = 1, \tag{2.1}$$

be proper in every component. Polynomial parts, if present, may be removed before applying the results below. Put

$$G = \gcd(Q, P_1, \dots, P_d), \qquad Q_* = Q/G, \qquad \mathbf{P}_* = \mathbf{P}/G, \tag{2.2}$$

and set $r = \deg Q_*$. Write

$$\mathbf{R}(z) = \sum_{n \geq 0} \mathbf{c}_n z^n, \qquad Q_*(z) = 1 + q_1 z + \cdots + q_r z^r.$$

The reciprocal recurrence polynomial is

$$a_*(x) = x^r + q_1 x^{r-1} + \cdots + q_r,$$

and

$$\mathbf{c}_{n+r} + q_1 \mathbf{c}_{n+r-1} + \cdots + q_r \mathbf{c}_n = 0 \qquad (n \geq 0). \tag{2.3}$$

For integers $L, K \geq r$, define

$$H_\nu = [\mathbf{c}_{i+j+\nu}]_{\substack{0 \leq i < L \\ 0 \leq j < K}} \in \mathbb{C}^{dL \times K}, \qquad \nu = 0, 1. \tag{2.4}$$

Each displayed entry in (2.4) is a d-vector stacked vertically.

Theorem 2.1 (Exact classification)**.** *For the proper vector rational function (2.1), the following statements hold.*

(1) $\operatorname{rank} H_0 = r$.
(2) *Some r-by-r submatrix $G_0 = H_0[I, J]$ is invertible. For the matching shifted submatrix $G_1 = H_1[I, J]$,*

$$A_{I,J} = G_1 G_0^{-1} \tag{2.5}$$

is similar to a cyclic minimal realization and

$$\det(I - z A_{I,J}) = Q_*(z). \tag{2.6}$$

(3) *If*

$$Q_*(z) = \prod_{j=1}^{s} (1 - \lambda_j z)^{m_j}$$

with distinct λ_j, then $A_{I,J}$ has one Jordan block of size m_j at λ_j.
(4) *For every divisor $S \mid Q$, the following are equivalent:*

$$S \mid G, \qquad S \mid P_\ell \text{ for every } \ell, \qquad S \text{ is absent from the minimal recurrence},$$

and

$$\frac{\mathbf{P}}{Q} = \frac{\mathbf{P}/S}{Q/S}.$$

Consequently the complete Froissart divisor is G, including arbitrary finite mixed factors, repeated factors, and excess multiplicity colliding with a true pole.

Proof. Choose a minimal one-input, d-output realization

$$\mathbf{c}_n = CA^n b, \qquad \det(I - zA) = Q_*(z). \tag{2.7}$$

Minimality means that (A, b) is controllable and (C, A) is observable. Define

$$\mathcal{O}_L = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix}, \qquad \mathcal{K}_K = \begin{bmatrix} b & Ab & \cdots & A^{K-1}b \end{bmatrix}.$$

Then

$$H_0 = \mathcal{O}_L \mathcal{K}_K, \qquad H_1 = \mathcal{O}_L A \mathcal{K}_K. \tag{2.8}$$

For $L, K \geq r$, both factors in the first identity have rank r; hence rank $H_0 = r$.

Let G_0 be any invertible r-by-r submatrix, and write

$$X = \mathcal{O}_L[I, :], \qquad Y = \mathcal{K}_K[:, J].$$

Then $G_0 = XY$ and $G_1 = XAY$. Invertibility of G_0 forces X and Y to be invertible, so

$$G_1 G_0^{-1} = XAX^{-1}.$$

The realization is single-input and controllable, hence cyclic. Its minimal and characteristic polynomials coincide, proving (2.6) and the one-block assertion.

The last equivalences are ordinary polynomial divisibility applied simultaneously to all numerators. Removing such a divisor leaves the same rational vector, and every factor removed from Q is absent from the minimal recurrence (2.3). Conversely, a denominator factor absent from the minimal recurrence belongs to $Q/Q_* = G$. □

3. Confluent jets and residue geometry

Let

$$S(z) = \prod_{j=1}^{s} (z - \zeta_j)^{\mu_j}, \qquad \mu = \sum_{j=1}^{s} \mu_j, \tag{3.1}$$

with distinct ζ_j. For a polynomial P, define the confluent Hermite jet

$$\mathcal{J}_S(P) = \left(\frac{P^{(k)}(\zeta_j)}{k!} \right)_{\substack{1 \leq j \leq s \\ 0 \leq k < \mu_j}}. \tag{3.2}$$

On polynomials of degree below μ, its matrix in ascending coefficients is

$$\Phi_S[(j,k), n] = \begin{cases} \binom{n}{k} \zeta_j^{n-k}, & n \geq k, \\ 0, & n < k. \end{cases} \tag{3.3}$$

Lemma 3.1 (Jet and principal-part geometry)**.** *The matrix Φ_S is invertible. If $R = P \bmod S$, then*

$$\sigma_{\min}(\Phi_S)\|\operatorname{coef} R\|_2 \le \|\mathcal{J}_S(P)\|_2 \le \sigma_{\max}(\Phi_S)\|\operatorname{coef} R\|_2. \tag{3.4}$$

In particular, $S \mid P$ if and only if $\mathcal{J}_S(P) = 0$.

Suppose also that $Q = ST$. The numerator jets at the roots of S and the principal-part coefficient jets of P/Q are related by an invertible block lower triangular matrix $\mathcal{T}_{S,Q}$. Therefore

$$\sigma_{\min}(\mathcal{T}_{S,Q})\|\mathcal{J}_S(\mathbf{P})\|_F \le \|\operatorname{PP}_S(\mathbf{P}/Q)\|_F \le \|\mathcal{T}_{S,Q}\|_2\|\mathcal{J}_S(\mathbf{P})\|_F. \tag{3.5}$$

Proof. The confluent Hermite interpolation problem at the distinct nodes ζ_j has a unique solution of degree below μ, so Φ_S is invertible. Since P and its remainder R have identical jets through the prescribed orders, applying the extremal singular-value bounds to $\Phi_S \operatorname{coef} R$ gives (3.4).

Locally at ζ_j, write

$$Q(z) = (z - \zeta_j)^{\mu_j} u_j(z), \qquad u_j(\zeta_j) \neq 0.$$

Multiplication of the numerator Taylor jet by the truncated Taylor series of $1/u_j$ is an invertible lower triangular map whose diagonal entries are $u_j(\zeta_j)^{-1}$. Taking the block diagonal sum over the roots and all channels proves (3.5). ☐

Remark 3.2. For Hermite–Padé data the Frobenius norm in (3.5) includes every numerator channel. A factor that cancels from one component but not another is genuine for the common denominator.

4. Finite perturbations and the recovered pencil

Let a rational pole head of minimal order r have coefficients

$$\mathbf{c}_n^R = CA^n b.$$

Suppose the observed d-channel coefficients satisfy

$$\widetilde{\mathbf{c}}_n = \mathbf{c}_n^R + \mathbf{e}_n, \qquad \|\mathbf{e}_n\|_2 \le \epsilon_n. \tag{4.1}$$

This includes a meromorphic-plus-analytic decomposition

$$\mathbf{F}(z) = \mathbf{R}(z) + \mathbf{a}(z), \qquad \sup_{|z| \le R} \|\mathbf{a}(z)\|_2 \le M.$$

If measurement error in coefficient n is at most δ_n, Cauchy's estimate permits

$$\epsilon_n = \delta_n + MR^{-n}. \tag{4.2}$$

For a window beginning at n_0, form

$$\widetilde{H}_\nu = [\widetilde{\mathbf{c}}_{n_0+i+j+\nu}]_{\substack{0 \le i < L \\ 0 \le j < K}},$$

and define

$$\eta_\nu = \left(\sum_{i=0}^{L-1} \sum_{j=0}^{K-1} \epsilon^2_{n_0+i+j+\nu} \right)^{1/2}. \tag{4.3}$$

Then

$$\|\widetilde{H}_\nu - H_\nu^R\|_2 \le \eta_\nu. \tag{4.4}$$

Definition 4.1 (Separated rank class)**.** For fixed data bounds and window, the rational head belongs to the separated order-r class if

$$\operatorname{rank} H_0^R = r, \qquad \sigma_r(H_0^R) > 3\eta_0. \tag{4.5}$$

The condition is an observability requirement. Indeed,

$$H_0^R = \mathcal{O}_L A^{n_0} \mathcal{K}_K$$

and hence

$$\sigma_r(H_0^R) \ge \sigma_{\min}(\mathcal{O}_L)\sigma_{\min}(A^{n_0}\mathcal{K}_K). \tag{4.6}$$

In a Jordan basis, the two factors contain confluent Vandermonde geometry and the highest nonzero residue jets.

Theorem 4.2 (Rank and pivot recovery)**.** *Assume the separated rank condition (4.5). Then exactly r singular values of $\widetilde{H}_0$ exceed $2\eta_0$, and every remaining singular value is at most η_0.*

Choose a matching r-by-r pivot

$$\widetilde{G}_\nu = \widetilde{H}_\nu[I, J],$$

put $\widetilde{\sigma} = \sigma_{\min}(\widetilde{G}_0)$, and assume $\widetilde{\sigma} > \eta_0$. Define

$$\widehat{A} = \widetilde{G}_1 \widetilde{G}_0^{-1}, \qquad e_A = \frac{\eta_1 + \eta_0\|\widehat{A}\|_2}{\widetilde{\sigma} - \eta_0}. \tag{4.7}$$

For the exact matching pivot realization $A_{I,J}$,

$$\|\widehat{A} - A_{I,J}\|_2 \le e_A. \tag{4.8}$$

Proof. Weyl's singular-value inequalities and $\operatorname{rank} H_0^R = r$ give

$$\sigma_j(\widetilde{H}_0) \ge \sigma_j(H_0^R) - \eta_0 > 2\eta_0 \quad (1 \le j \le r),$$

and

$$\sigma_j(\widetilde{H}_0) \le \eta_0 \quad (j > r).$$

Let $G_\nu = H_\nu^R[I, J]$ and $E_\nu = \widetilde{G}_\nu - G_\nu$. The submatrix error is at most η_ν, so

$$\sigma_{\min}(G_0) \ge \widetilde{\sigma} - \eta_0 > 0.$$

The exact pivot therefore exists. Using

$$\widehat{A}\widetilde{G}_0 = \widetilde{G}_1, \qquad A_{I,J}G_0 = G_1,$$

we obtain

$$(\widehat{A} - A_{I,J})G_0 = E_1 - \widehat{A}E_0.$$

Right multiplication by G_0^{-1} gives (4.7)–(4.8). ∎

Remark 4.3 (What the rank statement certifies)**.** The theorem is uniform on the separated class (4.5). The empty observed band $(\eta_0, 2\eta_0]$ is a useful fail-closed diagnostic, but finite noisy data alone cannot prove that an additional exact singular value does not lie below the resolution floor. Section 8 shows that no method can remove this qualification.

Corollary 4.4 (Denominator coefficient ball)**.** *Set*

$$\widehat{Q}_*(z) = \det(I - z\widehat{A}), \qquad M_A = \|\widehat{A}\|_2 + e_A.$$

Then

$$\|\operatorname{coef}(\widehat{Q}_* - Q_*)\|_1 \le \delta_Q := re_A(1 + M_A)^{r-1}. \tag{4.9}$$

Proof. The coefficient of z^k in $\det(I - zA)$, up to sign, is the sum of the k-by-k principal minors of A. Telescoping the columns of each determinant bounds the change of one such minor by $ke_A M_A^{k-1}$. Summing gives

$$\sum_{k=1}^{r} \binom{r}{k} ke_A M_A^{k-1} = re_A(1 + M_A)^{r-1}.$$

□

5. Contour-stable multiplicity

Individual roots of a nonnormal or confluent realization can be highly unstable. We therefore use Riesz projectors. Let Γ be a positively oriented rectifiable contour avoiding $\operatorname{spec}(\widehat{A})$, and put

$$\widehat{\kappa}_\Gamma = \max_{z \in \Gamma} \|(zI - \widehat{A})^{-1}\|_2. \tag{5.1}$$

Theorem 5.1 (Riesz-stable cluster multiplicity)**.** *If*

$$e_A \widehat{\kappa}_\Gamma < 1, \tag{5.2}$$

then $\widehat{A}$ and $A_{I,J}$ have the same total algebraic multiplicity inside Γ. Their Riesz projectors satisfy

$$\|\widehat{\mathcal{P}}_\Gamma - \mathcal{P}_\Gamma\|_2 \le \frac{\operatorname{len}(\Gamma)}{2\pi} \frac{e_A \widehat{\kappa}_\Gamma^2}{1 - e_A \widehat{\kappa}_\Gamma}. \tag{5.3}$$

Proof. Write $E = A_{I,J} - \widehat{A}$. Along $A_t = \widehat{A} + tE$,

$$zI - A_t = (zI - \widehat{A}) \left[I - tE(zI - \widehat{A})^{-1} \right].$$

Condition (5.2) makes the bracket invertible for every $t \in [0, 1]$ and $z \in \Gamma$. Thus no eigenvalue crosses the contour, and the projector rank is constant. The resolvent identity and the Neumann bound give

$$\|(zI - A_{I,J})^{-1} - (zI - \widehat{A})^{-1}\|_2 \le \frac{e_A \widehat{\kappa}_\Gamma^2}{1 - e_A \widehat{\kappa}_\Gamma}.$$

Integrating around Γ proves (5.3). □

Proposition 5.2 (Finite circle sampling)**.** *Let Γ be a circle of radius R_Γ, sampled at N equally spaced points. If s_* is the least sampled value of $\sigma_{\min}(zI - \widehat{A})$, then*

$$s_* > 2R_\Gamma \sin\frac{\pi}{2N} \implies \widehat{\kappa}_\Gamma \le \Big(s_* - 2R_\Gamma \sin\frac{\pi}{2N}\Big)^{-1}. \tag{5.4}$$

Proof. Every contour point is within chord distance $2R_\Gamma \sin(\pi/(2N))$ of a sample. The least singular value is 1-Lipschitz in the matrix argument, so subtracting that chord reserve gives a positive continuous lower bound. □

6. Nominal excess and shell classification

Let Q_N be a nominal denominator of degree N, possibly $N > r$, and let

$$Q_N^\vee(x) = x^N Q_N(x^{-1})$$

be its reciprocal polynomial. Choose pairwise disjoint contours Γ_a in the reciprocal-pole plane. Let n_a be the nominal algebraic zero count of $Q_N^\vee$ inside Γ_a, and let m_a be the Riesz count of $\widehat{A}$. Under Theorem 5.1, define the stable cluster excess

$$f_a = n_a - m_a. \tag{6.1}$$

Proposition 6.1 (Certified nominal count)**.** *Suppose an observed reciprocal denominator $\widehat{D}(z) = \sum_k \widehat{d}_k z^k$ has coefficient bounds*

$$|\widehat{d}_k - d_k| \le \beta_k$$

relative to an exact denominator D. If

$$\min_{z\in\Gamma} |\widehat{D}(z)| > \max_{z\in\Gamma} \sum_k \beta_k |z|^k, \tag{6.2}$$

then D and $\widehat{D}$ have the same algebraic zero count inside Γ.

For a circle centered at c with radius R_Γ, the left side of (6.2) has the finite lower bound

$$\min_{\text{samples}} |\widehat{D}| - 2R_\Gamma \sin\frac{\pi}{2N} \sum_{k\ge 1} k|\widehat{d}_k|(|c| + R_\Gamma)^{k-1}. \tag{6.3}$$

Proof. Condition (6.2) is Rouché's inequality on the contour. For (6.3), bound the derivative of $\widehat{D}$ on the closed disk containing the circle and multiply by the nearest-sample chord distance. □

A positive f_a states that the nominal denominator has excess multiplicity in the cluster. It does not identify which of several split nominal roots is removable. To certify a local factor, return to the original z-plane. Let S be as in (3.1), with its reciprocal roots supported on the certified contours, and define

$$\rho_S = \|\mathcal{J}_S(\mathbf{P})\|_F. \tag{6.4}$$

Suppose $|\widehat{\rho}_S - \rho_S| \le e_{\text{jet}}$. For fixed roots and a numerator coefficient bound $\|\Delta \mathbf{p}\|_F \le \delta_P$, one may take

$$e_{\text{jet}} \le \|\Phi_S\|_2 \delta_P. \tag{6.5}$$

If only root disks $|\zeta_j - \widehat{\zeta}_j| \le d_j$ are known, then for $n > k$

$$|\Delta\Phi[(j,k),n]| \le \binom{n}{k}(n-k)(|\widehat{\zeta}_j| + d_j)^{n-k-1} d_j. \tag{6.6}$$

Let δ_Φ be the Frobenius norm of the entrywise bounds. Then

$$e_{\text{jet}} \le \|\widehat{\Phi}_S\|_2 \delta_P + \delta_\Phi(\|\widehat{\mathbf{p}}\|_F + \delta_P). \tag{6.7}$$

Theorem 6.2 (No-false-removal rule)**.** *Choose* $0 \le \tau_{\text{rem}} < \tau_{\text{true}}$*. For a candidate divisor* S*, let* $f(S)$ *be the sum of the positive excess counts on its supporting contours. Use the rule*

$$\begin{array}{lll} f(S) \ge \deg S, \quad \widehat{\rho}_S + e_{\text{jet}} \le \tau_{\text{rem}} & \Longrightarrow & \text{removable}, \\ \widehat{\rho}_S - e_{\text{jet}} \ge \tau_{\text{true}} & \Longrightarrow & \text{genuine}, \\ \textit{otherwise} & \Longrightarrow & \text{unresolved}. \end{array} \tag{6.8}$$

Assume that every candidate whose removal would delete a multiplicity of the minimal denominator satisfies

$$\rho_S \ge \tau_{\text{true}}. \tag{6.9}$$

Then (6.8) has zero false removals.

Proof. Suppose a candidate classified as removable would delete a minimal denominator multiplicity. By (6.9), $\rho_S \ge \tau_{\text{true}}$. The certified jet error implies

$$\widehat{\rho}_S + e_{\text{jet}} \ge \rho_S \ge \tau_{\text{true}},$$

contradicting the removable inequality and $\tau_{\text{rem}} < \tau_{\text{true}}$. The excess condition additionally prevents local removal when the nominal denominator has not supplied enough certified excess degree. ☐

Remark 6.3. The genuine margin is not cosmetic. Without it, a true pole with a sub-noise principal part is indistinguishable from cancellation. The correct output in the gap between the two thresholds is unresolved.

7. Certified removal

7.1. **Local common-factor division.** Suppose numerical polynomial division gives

$$Q = ST + D, \qquad \mathbf{P} = S\mathbf{U} + \mathbf{R}_{\text{rem}}, \qquad \deg D, \deg R_{\text{rem},\ell} < \mu. \tag{7.1}$$

Theorem 7.1 (Local division error)**.** *The exact identity*

$$\frac{\mathbf{P}}{Q} - \frac{\mathbf{U}}{T} = \frac{\mathbf{R}_{\text{rem}}T - \mathbf{U}D}{QT} \tag{7.2}$$

holds. Let $K \subset \mathbb{C}$ be compact and set

$$v_\mu(K) = \sup_{z \in K} \left(\sum_{n=0}^{\mu-1} |z|^{2n} \right)^{1/2},$$

$$m_Q = \inf_K |Q|, \qquad m_T = \inf_K |T|, \qquad M_U = \sup_K \|\mathbf{U}\|_2.$$

If $m_Q, m_T > 0$, then

$$\left\| \frac{\mathbf{P}}{Q} - \frac{\mathbf{U}}{T} \right\|_K \le \frac{v_\mu \| \operatorname{coef} \mathbf{R}_{\mathrm{rem}} \|_F}{m_Q} + \frac{M_U v_\mu \| \operatorname{coef} D \|_2}{m_Q m_T}. \tag{7.3}$$

When $D = 0$,

$$\| \operatorname{coef} \mathbf{R}_{\mathrm{rem}} \|_F \le \| \Phi_S^{-1} \|_2 \rho_S. \tag{7.4}$$

Proof. Expanding the numerator of the difference gives

$$(S\mathbf{U} + \mathbf{R}_{\mathrm{rem}})T - \mathbf{U}(ST + D) = \mathbf{R}_{\mathrm{rem}} T - \mathbf{U}D,$$

which proves (7.2). Every polynomial remainder of degree below μ satisfies

$$|R(z)| \le v_\mu(K) \| \operatorname{coef} R \|_2$$

on K. Applying this inequality to both terms in (7.2) proves (7.3). Equation (7.4) is Lemma 3.1. □

If S is an exact Froissart factor, both division remainders vanish and the rational function is unchanged. For a near-cancellation, every nonzero remainder is charged explicitly.

7.2. **Minimal-model reconstruction.** Local root assignment may be ill-conditioned even when a contour excess is stable. The alternative is to reconstruct the minimal rational head directly. Write

$$\widehat{Q}_*(z) = \sum_{k=0}^{r} \widehat{q}_k z^k$$

and define, for $0 \le n < r$,

$$\widehat{\mathbf{p}}_n = \sum_{k=0}^{n} \widehat{q}_k \widetilde{\mathbf{c}}_{n-k}. \tag{7.5}$$

Set

$$E_{<r} = \sum_{n=0}^{r-1} \epsilon_n, \qquad \widetilde{C}_{<r} = \sum_{n=0}^{r-1} \|\widetilde{\mathbf{c}}_n\|_2,$$

and

$$\delta_P = (\|\widehat{Q}_*\|_1 + \delta_Q) E_{<r} + \delta_Q \widetilde{C}_{<r}. \tag{7.6}$$

Theorem 7.2 (Root-pairing-free reconstruction)**.** *The reconstructed numerator obeys*

$$\sum_{n=0}^{r-1} \|\widehat{\mathbf{p}}_n - \mathbf{p}_n\|_2 \le \delta_P. \tag{7.7}$$

Let $K \subset \mathbb{C}$ be compact, $R_K = \sup_K |z|$, and put

$$\overline{\delta}_Q = \delta_Q \max_{0\le k\le r} R_K^k, \qquad \overline{\delta}_P = \delta_P \max_{0\le k< r} R_K^k.$$

If

$$\widehat{m}_Q := \inf_K |\widehat{Q}_*| > \overline{\delta}_Q, \tag{7.8}$$

then, with $\widehat{M}_P = \sup_K \|\widehat{\mathbf{P}}_*\|_2$,

$$\left\| \frac{\widehat{\mathbf{P}}_*}{\widehat{Q}_*} - \frac{\mathbf{P}_*}{Q_*} \right\|_K \le \frac{\overline{\delta}_P}{\widehat{m}_Q} + \frac{(\widehat{M}_P + \overline{\delta}_P)\overline{\delta}_Q}{\widehat{m}_Q(\widehat{m}_Q - \overline{\delta}_Q)}. \tag{7.9}$$

Proof. For each coefficient, split the convolution error as

$$\widehat{Q}_*\widetilde{\mathbf{c}} - Q_*\mathbf{c}^R = Q_*(\widetilde{\mathbf{c}} - \mathbf{c}^R) + (\widehat{Q}_* - Q_*)\widetilde{\mathbf{c}}.$$

Summing the resulting convolution bounds over $0 \le n < r$, and using $\|Q_*\|_1 \le \|\widehat{Q}_*\|_1 + \delta_Q$, proves (7.7).

On K, coefficient ℓ^1 bounds give

$$\|\widehat{\mathbf{P}}_* - \mathbf{P}_*\|_2 \le \overline{\delta}_P, \qquad |\widehat{Q}_* - Q_*| \le \overline{\delta}_Q.$$

Thus $|Q_*| \ge \widehat{m}_Q - \overline{\delta}_Q > 0$. Use

$$\frac{\widehat{\mathbf{P}}_*}{\widehat{Q}_*} - \frac{\mathbf{P}_*}{Q_*} = \frac{\widehat{\mathbf{P}}_* - \mathbf{P}_*}{\widehat{Q}_*} + \frac{\mathbf{P}_*(Q_* - \widehat{Q}_*)}{\widehat{Q}_* Q_*}$$

and $\|\mathbf{P}_*\|_2 \le \widehat{M}_P + \overline{\delta}_P$ to obtain (7.9). □

Corollary 7.3 (Full meromorphic input)**.** *If $\mathbf{F} = \mathbf{R} + \mathbf{a}$ and K lies in the analytic-tail domain, then the right side of (7.9) plus $\|\mathbf{a}\|_K$ bounds the error between the reconstructed rational head and $\mathbf{F}$.*

7.3. **The generalized shell notch.** For $S(z) = \sum_{k=0}^{\mu} s_k z^k$, define

$$(\mathcal{N}_S\mathbf{c})_n = \sum_{k=0}^{\min(\mu,n)} s_k \mathbf{c}_{n-k}. \tag{7.10}$$

Proposition 7.4 (Shell notch)**.** *The transformation (7.10) multiplies the generating function by S. It annihilates every reciprocal-pole Jordan block rooted in S after the initial boundary. On every other block it acts invertibly. If $\|\mathbf{e}_n\|_2 \le \epsilon_n$, then*

$$\|(\mathcal{N}_S\mathbf{e})_n\|_2 \le \sum_k |s_k|\epsilon_{n-k}, \tag{7.11}$$

and in every translation-invariant sequence norm,

$$\|\mathcal{N}_S \mathbf{e}\| \leq \|S\|_1 \|\mathbf{e}\|. \tag{7.12}$$

The factor $\|S\|_1$ is sharp without further noise assumptions.

Proof. The convolution identity is immediate. On a mode generated by a reciprocal Jordan block J, the multiplier is the triangular matrix $S(J^{-1})$. It vanishes on the rooted generalized eigenspace with the prescribed multiplicity and is invertible when S does not vanish there. Equations (7.11)–(7.12) are the triangle inequality and Young's convolution inequality. At a selected index, choose error phases aligned with the coefficients s_k to attain the coefficientwise sum. □

Remark 7.5. The notch is a pole filter and diagnostic. It is not representation-preserving common-factor removal: multiplication by S changes the represented function, whereas division in Theorem 7.1 removes a common factor.

8. Information limits

Theorem 8.1 (Sharp finite-data limitations)**.** *Fix observed indices $0 \leq n \leq M$ and an error radius $\epsilon > 0$.*

(1) *For every $\lambda \in \mathbb{C}$, there is a nonzero a such that the zero sequence and $c_n = a\lambda^n$ have distance at most ϵ on the observed indices. No method can uniformly decide whether this pole mode is present throughout both admissible error balls.*
(2) *For every $m \geq 2$, $\lambda \in \mathbb{C}$, and $\delta > 0$, the Jordan block $J_m(\lambda)$ is within operator-norm distance δ of a matrix with m distinct eigenvalues. Exact Jordan partitions are not stable under arbitrary norm-bounded perturbations.*

Proof. Choose $a \neq 0$ so that

$$|a| \max_{0 \leq n \leq M} |\lambda|^n \leq \epsilon.$$

The two finite coefficient vectors then have distance at most ϵ, so their radius-ϵ data sets intersect, although one model has a pole mode and the other does not.

For the second statement, choose distinct $\delta_1, \ldots, \delta_m$ with $|\delta_j| < \delta$. The upper triangular matrix

$$J_m(\lambda) + \operatorname{diag}(\delta_1, \ldots, \delta_m)$$

has m distinct eigenvalues and lies within δ of $J_m(\lambda)$. □

Theorem 8.1 explains both qualifications in the positive theory. The rank result needs a residue/observability floor, and the perturbative spectral output is a contour count. Exact arithmetic still recovers exact multiplicities through Theorem 2.1.

9. Fail-closed recovery procedure

The preceding proofs give the following finite procedure.

(1) Form $\widetilde{H}_0, \widetilde{H}_1$ and the rigorous bounds η_0, η_1 from (4.3).
(2) Count singular values above $2\eta_0$. Stop if any discarded singular value lies in $(\eta_0, 2\eta_0]$. Interpret the result only on the separated class of Definition 4.1.
(3) Select a rank-revealing square pivot. Stop if $\widetilde{\sigma} \leq \eta_0$. Compute $\widehat{A}$ and e_A.
(4) Certify requested reciprocal-pole contours using Proposition 5.2 and Theorem 5.1. Record total algebraic multiplicities, not split root labels.
(5) If a nominal denominator is supplied, certify its contour counts by Proposition 6.1 and compute the excess counts (6.1).
(6) For every proposed local factor, compute all-channel confluent jets and their coefficient/root uncertainty. Apply Theorem 6.2; never remove an unresolved factor.
(7) Either use local division and (7.3), or reconstruct the minimal model and use (7.9). Stop if the required denominator floor is nonpositive.
(8) If coefficient-level pole filtering is useful, apply the notch and report its exact error amplification.

The procedure handles scalar Padé approximation by taking $d = 1$, and common-denominator Hermite–Padé approximation for arbitrary finite d. Equal modulus, mixed real/nonreal shells, repeated factors, and collided true/spurious multiplicities require no separate cases.

10. Numerical illustrations

The accompanying software checks the formulas on exact and perturbed test families. These calculations use double-precision linear algebra and are regression evidence, not formal interval certificates. Table 1 reports one two-channel example with minimal denominator

$$Q_*(z) = (1 - 1.2z)^2(1 + 0.7z).$$

The perturbed coefficients have componentwise magnitude 10^{-9}, and a 5-by-5 block-Hankel window is used.

Table 1. Representative perturbation bounds and observed errors.

Quantity	observed error	certified upper bound
realization, spectral norm	$1.1895 \cdot 10^{-9}$	$6.6784 \cdot 10^{-9}$
denominator coefficients, ℓ^1	$1.1514 \cdot 10^{-9}$	$1.5742 \cdot 10^{-7}$
near-deflation function error	$5.2700 \cdot 10^{-7}$	$1.5848 \cdot 10^{-6}$

Here $\eta_0 = \eta_1 = 8.4853 \cdot 10^{-10}$, the observed pivot has smallest singular value 0.35615, and the sampled contour gives $\widehat{\kappa}_\Gamma \leq 20.314$. The Riesz-projector error bound is $5.512 \cdot 10^{-7}$. Separate exact tests include a degree-six mixed confluent common factor, a two-channel cancellation invisible to either channel alone, and excess multiplicity colliding with a genuine double pole.

The sharpness tests also realize the two limitations in Theorem 8.1: a pole mode whose largest observed coefficient is exactly 10^{-8} at a 10^{-8} noise floor, and a 2-by-2 Jordan block split by an operator-norm perturbation of 10^{-9}.

11. Relation to existing approaches

The exact block-Hankel factorization is a finite minimal-realization statement, and the local conditioning quantities are consonant with confluent Prony analysis [2]. The distinction from simple Froissart-doublet bounds [3, 4] is that no root pairing, simplicity, or scalar residue is assumed. The nominal excess is counted by contours, and a separate confluent all-channel coupling determines whether a specific divisor may be removed.

The distinction from SVD and matrix-pencil algorithms [7, 10, 6] is the explicit fail-closed perturbation chain: coefficient errors to Hankel errors, pivot errors, denominator coefficient errors, continuous resolvent bounds, algebraic counts, jet margins, and finally a uniform rational-function bound. The negative result of Mascarenhas [8] is consistent with this chain: conditioning of one defining Padé matrix is not used as a substitute for coprimeness, observability, or contour separation.

Approximate-GCD methods [9] can be used to propose the local factor S, but the theorem does not infer safety from an approximate GCD score alone. The factor must also be supported by excess contour multiplicity and an all-channel jet margin. Conversely, if a cluster excess is stable but no local factor is stable, minimal-model reconstruction remains available.

12. Limitations

The theory is finite and nonasymptotic, but not assumption-free.

- Coefficient-error bounds must be rigorous for the application at hand.
- The exact rational head must lie in the separated observability class if its full order is to be identified.
- Every spectral contour needs a resolvent margin, and every nominal count needs a Rouché margin.
- Zero false removals require a lower coupling threshold for every true pole multiplicity that must be preserved.
- Uniform function bounds apply only on compact sets with positive denominator floors.

- Total algebraic multiplicity is stable; individual Jordan blocks and split repeated roots are not.
- The results concern finite rational pole heads plus bounded analytic tails. They do not classify asymptotic Froissart strings modeling a branch cut.

The constants have been chosen for transparent proofs rather than optimality. Better rank-revealing pivots, structured perturbation theory, and sharper determinant coefficient bounds may materially improve them.

13. Data and code availability

No external data are used. The publication package includes the complete LaTeX source, a deterministic source archive, and a self-contained reproducibility archive containing the numerical implementation, focused tests, pinned example outputs, and exact run commands. The computations reproduce the numerical illustrations; the theorems themselves are proved analytically in the text.

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