Endpoint-flat arithmetic approximants to the Riemann Ξ-function with quantitative real strip tails

Kenan

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Abstract

We construct compact arithmetic approximants to the Riemann Ξ-function whose zeros are eventually real and simple in every fixed closed substrip of the critical strip. The mechanism is an exact Poisson decomposition of an endpoint-flat arithmetic kernel into a universal exponential endpoint jet and Fourier aliases. The endpoint jet resums to the characteristic function of a positive Neumann–Robin Sturm–Liouville problem. Repeated application of $B = -\partial_y^2 + 1/4$ gives an explicit cellwise Rouché criterion in which the transformed remainder and every endpoint defect are visible.

For a compactly flattened Gaussian–Hermite carrier we obtain convergence to $\Xi/(2\pi)$, uniformly in the entire real direction on every closed substrip, and an $O(\log c)$ nonreal core at arithmetic cutoff c. Applying the shifted Euler heat semigroup improves the core to $O(\sqrt{\log c})$. More generally, for each fixed integer $k \geq 1$ and $\alpha > 0$, a logarithmic Gevrey compactification of $\exp(-\alpha \mathcal{Z}^k)$, where $\mathcal{Z} = -(x\partial_x + 1/2)^2$, produces endpoint-flat sources whose transforms converge to

$$\frac{\Xi(z)}{2\pi} e^{-\alpha z^{2k}}$$

and whose nonreal zeros in a fixed substrip are confined to $O((\log c)^{1/(2k)})$. Thus any prescribed positive fixed power of $\log c$ is attainable. The multiplier is entire and zero-free, but the core still diverges; the results do not prove the Riemann hypothesis.

Keywords. Riemann Xi-function; arithmetic approximants; endpoint flatness; Poisson summation; Dini functions; real zeros; Mellin transform; Gevrey cutoff.

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1 Introduction

Write

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \Xi(z) = \xi\left(\frac{1}{2} + iz\right).$$

The Riemann hypothesis is equivalent to the assertion that every zero of Ξ is real. Approximation alone does not make this formulation easier: locally uniform approximants inherit the zeros of the limit, real or not. The useful question is instead whether a natural arithmetic approximant admits a real-zero theorem outside a quantitatively controlled central region.

We answer that question for a family built from the arithmetic summation map

$$\mathcal{E}(f)(u) = u^{1/2} \sum_{n \geq 1} f(nu).$$

If a compact source has a nonzero value at its support endpoint, the resulting kernel has internal value jumps. Its remote zero pattern is then governed by a finite Dirichlet polynomial rather than by a single trigonometric wave. Exact endpoint flatness removes all such value jumps. Poisson summation then exposes a universal left-endpoint term

$$Ae^{y/2}$$

and a Fourier-alias remainder. After centering the support interval, the cosine transform of the exterior continuation of this exponential has numerator

$$D_L(z) = z \sin \frac{Lz}{2} - \frac{1}{2} \cos \frac{Lz}{2}.$$

This is a Dini function: its zeros are the eigenfrequencies of a positive Neumann–Robin problem and are therefore real and simple.

The first contribution of the paper is a quantitative version of this observation. Repeated integration by parts with $B = -\partial_y^2 + 1/4$ gives an exact decomposition

$$(z^2 + 1/4)^r F(z) = A(z^2 + 1/4)^{r-1} D_L(z) + \text{endpoint defects} + \widehat{B^r K}(z).$$

We state a fully explicit Rouché inequality on each Dini cell. Its normalization is uniform for strip heights approaching 1/2.

The second contribution is a sequence of source constructions. The classical Fourier-self-dual Gaussian–Hermite function

$$h(x) = x^2(2\pi x^2 - 3)e^{-\pi x^2}$$

satisfies

$$\zeta(s)M_h(s) = \frac{\xi(s)}{2\pi}.$$

A polynomial endpoint cutoff of order $2\lfloor \log \lambda \rfloor + 2$, together with a small Gaussian correction restoring zero integral, produces transforms converging to $\Xi/(2\pi)$ uniformly over the whole horizontal strip. Polynomial–Gaussian seminorm estimates yield a nonreal core of size $O(\log c)$, where $c = \lambda^2$.

The final contribution replaces polynomial–Gaussian differentiation by regularization in logarithmic coordinates. Under the unitary map

$$(\mathcal{U}f)(y) = e^{y/2} f(e^y),$$

the shifted Euler operator

$$\mathcal{Z} = -\left(x\partial_x + \frac{1}{2}\right)^2$$

becomes $-\partial_y^2$. The superheat filter $\mathcal{S}_{\alpha,k} = e^{-\alpha \mathcal{Z}^k}$ inserts the zero-free Mellin multiplier $e^{-\alpha z^{2k}}$. A scale-matched Gevrey cutoff makes the r-th transformed remainder grow only as $(Cr)^{r/k+O(1)}$. Comparing this with the Dini wave at $r \asymp \log \lambda$ gives the main result.

Theorem 1.1 (Main theorem). *Fix an integer $k \geq 1$, $\alpha > 0$, and $0 < Y < 1/2$. For every sufficiently large λ, with $c = \lambda^2 \in \mathbb{N}$, there is a real even source p_c, supported in $[-1, 1]$, having zero integral and vanishing to infinite order at ± 1, such that its exact finite arithmetic transform F_c obeys*

$$\sup_{\substack{\operatorname{Re} z \in \mathbb{R} \\ |\operatorname{Im} z| \leq Y}} \left| F_c(z) - \frac{\Xi(z)}{2\pi} e^{-\alpha z^{2k}} \right| = O_{\alpha,k,Y}\left(\lambda^{-3/2+Y}\right).$$

There is $K_{\alpha,k,Y} < \infty$ such that every zero of F_c with

$$|\operatorname{Im} z| \le Y, \qquad |\operatorname{Re} z| \ge K_{\alpha,k,Y} (\log \lambda)^{1/(2k)}$$

is real and simple.

Corollary 1.2. *For every $\varepsilon > 0$ and $0 < Y < 1/2$, one may choose a fixed k so that the nonreal zeros of the resulting approximants in $|\operatorname{Im} z| \le Y$ are confined to*

$$|\operatorname{Re} z| = O_{\varepsilon,\alpha,Y}((\log c)^{\varepsilon}).$$

The constants in theorem 1.1 may deteriorate rapidly with k; k is fixed before $c \to \infty$. No bounded-core or growing-k assertion is made. The divergence of the core is essential to the present argument.

The paper also records two comparison families. Fixed-order flattening of the prolate approximants in [4] preserves their locally uniform Ξ-limit and gives fixed-power real strip tails. The explicit Gaussian–Hermite family removes the prolate asymptotics and improves those tails to logarithmic scale. These comparisons clarify which estimates are structural and which belong to the final superheat construction.

The surrounding literature contains several distinct approximation mechanisms. Compact theta-kernel approximants were studied by Shi [10]; Mellin transforms of generalized Hermite functions by Coffey and Lettington [3]; and Fourier–Mellin structures around Poisson summation by Burnol [1, 2]. Romik's account [8] develops further Fourier and heat structures attached to ξ. The Euler semigroup itself is classical; see, for example, [9]. None of these results supplies the combined endpoint-flat arithmetic compactification and quantitative Dini-cell tail proved here.

2 The finite arithmetic transform

We use the Fourier convention

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} \,\mathrm{d}x.$$

Let q be real, even, and supported in $[-\lambda, \lambda]$. For $|y| \le \log \lambda$, define

$$k_{\lambda,q}(y) = e^{y/2} \sum_{n\ge 1} q(ne^y), \qquad F_{\lambda,q}(z) = \int_{-\log\lambda}^{\log\lambda} k_{\lambda,q}(y) \cos(zy) \,\mathrm{d}y.$$

The sums are finite at each y.

Lemma 2.1 (Whole-strip source estimate)**.** *Let $0 \le Y < 1/2$. If f is supported in $[-\lambda, \lambda]$, then*

$$\sup_{\substack{\operatorname{Re} z\in\mathbb{R}\\ |\operatorname{Im} z|\le Y}} |F_{\lambda,f}(z)| \le C_Y \lambda^{Y+1/2} \|f\|_{L^1(0,\lambda)}.$$

Proof. Write $z = t + iy_0$, with $|y_0| \le Y$. Since $|\cos(zy)| \le \cosh(Yy)$, it is enough to estimate the two exponentials in the hyperbolic cosine. For one of them, substitute $x = ne^y$, interchange the finite sum and integral, and obtain

$$\int_0^{\lambda} |f(x)| x^{-y_0-1/2} \sum_{x/\lambda\le n\le \lambda x} n^{y_0-1/2} \,\mathrm{d}x.$$

Uniformly for $-Y \le y_0 \le Y$,

$$\sum_{n\le N} n^{y_0-1/2} \le C_Y N^{y_0+1/2}.$$

The powers of x cancel, and enlarging the summation interval gives the asserted bound. The other exponential is identical. ☐

We next pass to the angular normalization. Let p be real, even, supported in $[-1,1]$, and define for $c>1$

$$K_{c,p}(x) = \frac{e^{x/2}}{\sqrt{c}} \sum_{1\le n<ce^{-x}} p\left(\frac{ne^x}{c}\right), \qquad 0\le x\le L := \log c.$$

If $c=\lambda^2$ and

$$p(v) = \lambda^{1/2} q(\lambda v),$$

then recentering $x = y+\log\lambda$ identifies the transform of $K_{c,p}$ with $F_{\lambda,q}$.

Proposition 2.2 (Exact Poisson splitting)**.** *Suppose $p(1)=0$, $\int_{-1}^{1} p = 0$, and p has enough regularity for Poisson summation. Then*

$$K_{c,p}(x) = -\frac{p(0)}{2\sqrt{c}} e^{x/2} + \sqrt{c}\, e^{-x/2} \sum_{\ell\ge 1} \widehat{p}(\ell c e^{-x}).$$

If

$$A_c = -\frac{p(0)}{2\sqrt{c}}, \qquad \mathcal{T} = -\left(\frac{1}{2} + \xi\partial_\xi\right),$$

then, whenever termwise differentiation is justified,

$$K_{c,p}^{(j)}(0) = 2^{-j} A_c + \sqrt{c} \sum_{\ell\ge 1} \mathcal{T}^j \widehat{p}(\xi)\Big|_{\xi=\ell c}.$$

Proof. Put $h = e^x/c$. Poisson summation gives

$$h\sum_{n\in\mathbb{Z}} p(nh) = \sum_{\ell\in\mathbb{Z}} \widehat{p}(\ell/h).$$

Evenness, $p(1)=0$, and $\widehat{p}(0)=0$ imply

$$\sum_{n\ge 1} p(nh) = h^{-1}\sum_{\ell\ge 1} \widehat{p}(\ell/h) - \frac{p(0)}{2}.$$

Multiplication by $e^{x/2}/\sqrt{c}$ proves the first identity. Differentiation gives the second. ☐

Remark 2.3. *The condition $p(1)=0$ is exact, not perturbative. In the finite arithmetic sum, a summand disappears when its argument reaches 1; the corresponding jump is proportional to $p(1)$. Any nonzero endpoint value restores the complete jump symbol, regardless of its size.*

3 Dini waves and real strip tails

We first give an elementary one-endpoint theorem.

Proposition 3.1 (Sine tail)**.** *Let K be real and continuous on $[0, L]$, piecewise C^2, with finite one-sided first derivatives and finite total derivative variation. If*

$$K(L) = 0, \qquad K(0) \neq 0,$$

then, for every $Y > 0$, all sufficiently remote zeros of

$$F(z) = \int_0^L K(x) \cos\left(z(x - L/2)\right) \mathrm{d}x$$

in $|\operatorname{Im} z| \leq Y$ are real and simple.

Proof. Two piecewise integrations by parts give

$$zF(z) = K(0) \sin \frac{Lz}{2} + R(z), \qquad |R(z)| \leq \frac{V_K \cosh(YL/2)}{|z|},$$

where V_K is the sum of the endpoint derivative magnitudes, all internal derivative jumps, and the L^1-norms of the second derivatives on the smooth cells.

The rectangles

$$\left|\operatorname{Re} \frac{Lz}{2} - n\pi\right| \leq \frac{\pi}{2}, \qquad |\operatorname{Im} z| \leq Y,$$

tile the strip. On their vertical sides $|\sin(Lz/2)| \geq 1$, and on their horizontal sides it is at least $\sinh(LY/2)$. Rouché's theorem gives one zero, counted with multiplicity, in each remote cell. The cell is invariant under conjugation, so its unique zero is real; multiplicity one makes it simple. □

The exact Poisson term in proposition 2.2 contains more information than its first boundary value. Extend $Ae^{x/2}$ to the left of the support interval. Its centered cosine transform is

$$-\int_{-\infty}^0 Ae^{x/2} \cos\left(z(x - L/2)\right) \mathrm{d}x = A \frac{D_L(z)}{z^2 + 1/4},$$

where

$$D_L(z) = z \sin \frac{Lz}{2} - \frac{1}{2} \cos \frac{Lz}{2}.$$

Lemma 3.2 (Dini zero geometry)**.** *All zeros of D_L are real and simple. There is exactly one positive zero in every interval*

$$\frac{(2n-1)\pi}{L} < x < \frac{(2n+1)\pi}{L}, \qquad n \geq 1.$$

Proof. With $a = L/2$, the equation $D_L(z) = 0$ is the characteristic equation for

$$-u'' = z^2 u, \qquad u'(0) = 0, \qquad u'(a) + \frac{1}{2} u(a) = 0.$$

The quadratic form is

$$\int_0^a |u'(x)|^2 \,\mathrm{d}x + \frac{1}{2}|u(a)|^2.$$

The problem is positive, selfadjoint, and separated, hence has positive simple eigenvalues. The location statement follows either from standard Sturm oscillation or directly from the equation $\tan(az) = 1/(2z)$. □

We now state the quantitative comparison used throughout the paper.

Theorem 3.3 (Higher-order Dini comparison)**.** *Let $r \geq 1$, $L > 0$, and*

$$B = -\partial_x^2 + \frac{1}{4}.$$

Suppose K is real, C^{2r-1} on $[0, L]$, piecewise C^{2r}, and

$$K^{(q)}(L) = 0, \qquad 0 \leq q \leq 2r - 1.$$

Put $U_j = B^j K$, $T(z) = z^2 + 1/4$, and

$$F(z) = \int_0^L K(x) \cos\left(z(x - L/2)\right) \, \mathrm{d}x.$$

For $A \neq 0$, define

$$\alpha_j = U_j(0) - A\mathbf{1}_{j=0}, \qquad \beta_j = U_j'(0) - \frac{A}{2}\mathbf{1}_{j=0}, \qquad \rho_r = \int_0^L |U_r(x)| \, \mathrm{d}x.$$

Then

$$\begin{aligned} T(z)^r F(z) = {} & A T(z)^{r-1} D_L(z) \\ & + \int_0^L U_r(x) \cos\left(z(x - L/2)\right) \, \mathrm{d}x \\ & + \sum_{j=0}^{r-1} T(z)^{r-1-j} \left(z\alpha_j \sin \frac{Lz}{2} - \beta_j \cos \frac{Lz}{2} \right). \end{aligned} \tag{3.1}$$

Fix $0 < Y < 1/2$, and put

$$x_n = \frac{(2n-1)\pi}{L}, \quad X_n = \frac{(2n+1)\pi}{L}, \quad C_Y = \cosh \frac{LY}{2}, \quad S_Y = \sinh \frac{LY}{2},$$

$$d_n = \min\left(x_n - \frac{S_Y}{2}, x_n S_Y - \frac{C_Y}{2} \right),$$

$$q_n = x_n^2 - Y^2 + \frac{1}{4}, \quad Q_n = X_n^2 + Y^2 + \frac{1}{4}, \quad Z_n = (X_n^2 + Y^2)^{1/2}.$$

If $d_n, q_n > 0$ and

$$|A| q_n^{r-1} d_n > C_Y \left[\rho_r + \sum_{j=0}^{r-1} Q_n^{r-1-j} \left(Z_n |\alpha_j| + |\beta_j| \right) \right], \tag{3.2}$$

then F has exactly one zero, counted with multiplicity, in

$$x_n \leq \operatorname{Re} z \leq X_n, \qquad |\operatorname{Im} z| \leq Y.$$

That zero is real and simple.

Proof. For a sufficiently smooth U, two integrations by parts give

$$\begin{aligned} T(z) \int_0^L U(x) \cos(z(x - L/2)) \, \mathrm{d}x = {} & \int_0^L BU(x) \cos(z(x - L/2)) \, \mathrm{d}x \\ & + z(U(0) + U(L)) \sin \frac{Lz}{2} \\ & + (U'(L) - U'(0)) \cos \frac{Lz}{2}. \end{aligned}$$

The endpoint assumptions imply $U_j(L) = U_j'(L) = 0$ for $j < r$. Iteration proves (3.1).

On the cell boundary, the nonreference terms are bounded by the right-hand side of (3.2). On a vertical side,

$$|D_L(z)| \geq x_n - \frac{S_Y}{2},$$

and on a horizontal side,

$$|D_L(z)| \geq x_n S_Y - \frac{C_Y}{2}.$$

Also $|T(z)| \geq \operatorname{Re} T(z) \geq q_n$. Thus (3.2) is precisely a strict Rouché inequality. The factors T^r have zeros only at $\pm i/2$, outside the cell. Lemma 3.2 gives one real simple reference zero. The perturbed cell has one zero counted with multiplicity; conjugation symmetry makes it real and simple. $\square$

Corollary 3.4 (Eventual coefficient test)**.** *In the setting of theorem 3.3, every sufficiently remote cell satisfies* (3.2) *if*

$$C_Y|\alpha_0| < |A| \min(1, S_Y).$$

Proof. As $n \to \infty$, the reference and the α_0-term are the only terms of degree $2r-1$ in x_n. Their leading coefficients are $|A|\min(1, S_Y)$ and $C_Y|\alpha_0|$, respectively. Every other term has lower degree. $\square$

Proposition 3.5 (Uniformity toward the strip edge)**.** *Fix $Y_0 \in (0, 1/2)$. Suppose a family in theorem 3.3 satisfies*

$$|A| \geq a_0 \lambda^{-5/2}, \qquad \rho_r \leq R,$$

and the endpoint defects obey the bounds used in sections 5 and 7. Then the resulting onset constants may be chosen simultaneously for every $Y \in [Y_0, 1/2)$.

Proof. After division by the common factor $\cosh(L|\operatorname{Im} z|/2)$, the Dini lower bound on a cell with left edge $x \geq 2$ contains

$$\delta_{x,Y} = \min\left(x, x\tanh(YL/2) - \frac{1}{2}\right).$$

Uniformly for $Y \geq Y_0$ and all sufficiently large L, $\delta_{x,Y} \geq x/4$. All remaining upper cell quantities are bounded by fixed multiples of their powers of x, independently of $Y < 1/2$. $\square$

4 The prolate comparison family

This section records how endpoint flattening interacts with the prolate approximants introduced in [4]. It is not needed for the proof of theorem 1.1, but it explains the passage from a qualitative tail theorem to the explicit Gaussian–Hermite construction.

Let $h_{0,\lambda}$ and $h_{4,\lambda}$ be the two fixed-index even prolate sources used in [4, Sec. 6]. Set

$$a_\lambda = \frac{\int h_{4,\lambda}}{\int h_{0,\lambda}}, \qquad h_\lambda = h_{4,\lambda} - a_\lambda h_{0,\lambda}.$$

Connes proves that, after a fixed nonzero real normalization, the Fourier–Mellin transforms of $\mathcal{E}(h_\lambda)$ converge to Ξ uniformly on every closed substrip of $|\operatorname{Im} z| < 1/2$.

Proposition 4.1 (Endpoint flattening preserves the prolate limit)**.** *Let m_λ be a positive integer and*

$$w_\lambda(x) = \left(1 - \frac{x^2}{\lambda^2}\right)^{m_\lambda}.$$

Choose a_{λ,m_λ} so that

$$\widetilde{h}_{\lambda,m_\lambda} = w_\lambda(h_{4,\lambda} - a_{\lambda,m_\lambda} h_{0,\lambda})$$

has zero integral. For each $Y < 1/2$,

$$\sup_{|\operatorname{Im} z| \le Y} \left| F_{\lambda,\widetilde{h}_{\lambda,m_\lambda}}(z) - F_{\lambda,h_\lambda}(z) \right| = O_Y(m_\lambda \lambda^{-3/2+Y}).$$

Consequently the flattened family has the same locally uniform Ξ-limit whenever

$$m_\lambda \lambda^{-3/2+Y} \longrightarrow 0 \quad \textit{for every fixed } Y < 1/2.$$

In particular, fixed order and $m_\lambda = O(\lambda)$ are admissible.

Proof. The fixed-mode parabolic-cylinder and endpoint Bessel expansions of Dunster [5] imply

$$\sup_{\lambda \ge \lambda_0} \int_0^\lambda (1 + x^2) |h_{\nu,\lambda}(x)| \, \mathrm{d}x < \infty, \qquad \nu = 0, 4,$$

and $\int h_{0,\lambda}$ tends to a nonzero Hermite integral. Since

$$0 \le 1 - (1 - s)^m \le ms, \qquad 0 \le s \le 1,$$

we have

$$\|(1 - w_\lambda) h_{\nu,\lambda}\|_1 = O(m_\lambda \lambda^{-2}).$$

The quotient defining a_{λ,m_λ} therefore differs from a_λ by the same order, and

$$\left\| \widetilde{h}_{\lambda,m_\lambda} - h_\lambda \right\|_1 = O(m_\lambda \lambda^{-2}).$$

Lemma 2.1 finishes the proof. □

The published, unflattened angular modes also have a useful exact endpoint ratio. Let $H_{\nu,c}$ be the even $L^2(-1,1)$-normalized angular modes at bandwidth $2\pi c$, oriented by positive finite Fourier eigenvalues $\chi_{\nu,c}$, and let

$$H_c = H_{4,c} - a_c H_{0,c}, \qquad e_c = H_c(1), \qquad r_c = \frac{c^{-1/2} \sum_{1 \le n < c} H_c(n/c)}{e_c}.$$

Proposition 4.2 (Published endpoint ratio)**.** *As the integer cutoff $c \to \infty$,*

$$r_c = 1 + O\left(\frac{\log c}{\sqrt{c}}\right).$$

Proof. Midpoint Poisson summation and the zero-integral condition give

$$c^{-1/2} \sum_{n=1}^{c-1} H_c(n/c) = \sum_{\ell \ge 1} \widetilde{H}_c(\ell) - \frac{H_c(0) + H_c(1)}{2\sqrt{c}},$$

where

$$\widetilde{H}_c = \chi_{4,c} H_{4,c} - a_c \chi_{0,c} H_{0,c}.$$

Dunster's radial expansion and integer connection phase imply, for fixed $\nu \in \{0, 4\}$,

$$\sum_{\ell \geq 2} \frac{H_{\nu,c}(\ell)}{H_{\nu,c}(1)} = O_\nu\left(\frac{\log c}{\sqrt{c}}\right).$$

For completeness, the range $2 \leq \ell < (2\pi c)^2$ follows from $|H_{\nu,c}(x)/H_{\nu,c}(1)| \ll (c^{1/2}x)^{-1}$. Beyond that point, variation of constants in the radial equation and the limiting integer phase give $O(\sqrt{c}/\ell^2)$, which is summable.

Fuchs's fixed-mode concentration asymptotic [6], together with its endpoint estimate, gives

$$H_{\nu,c}(1) = E_\nu e^{-2\pi c}(2\pi c)^{\nu/2+3/4}(1 + o(1)).$$

The fixed-mode Hermite limit and the exact finite-Fourier relation give

$$\frac{\widetilde{H}_c(1)}{H_c(1)} = 1 + e^{-4\pi c} c^{O(1)}, \qquad \frac{H_c(0)}{H_c(1)} = e^{-2\pi c} c^{O(1)}.$$

Substitution in the Poisson identity proves the result. □

Theorem 4.3 (Fixed-order prolate cores). *Fix $r \geq 2$, put $m = 2r + 2$, and flatten the two-mode prolate source as in proposition 4.1. For every $0 < Y < 1/2$, all zeros of the exact finite arithmetic transform satisfying*

$$|\operatorname{Im} z| \leq Y, \qquad |\operatorname{Re} z| \geq C_{r,Y} c^{5/(4(2r-1))}$$

are real and simple for all sufficiently large integer c. The transforms converge locally uniformly to Ξ. For $r = 4$, the exponent is $5/28$.

Proof. Put $c = \lambda^2$. The fixed-mode Hermite limit gives a nonzero center coefficient after flattening:

$$A_{c,m} = -\frac{m\Delta}{2}\lambda^{-5/2}(1 + o(1)), \qquad \Delta = \frac{2^{3/4}\sqrt{3}}{\pi^{3/4}} > 0.$$

Let $D = x\partial_x$, $\mathcal{L} = -D(D+1)$, and $g_{j,\lambda} = \mathcal{L}^j h_{\lambda,m}$. The arithmetic map intertwines $\mathcal{L}$ with B. The endpoint order of $g_{j,\lambda}$ is at least

$$s_j = m - 2j = 2(r - j) + 2.$$

The two final spare zeros make the zero extension of $g_{r,\lambda}$ C^1, so

$$\|\widehat{g}_{r,\lambda}\|_1 \leq C(\|g_{r,\lambda}\|_1 + \left\|g''_{r,\lambda}\right\|_1).$$

The fixed-seminorm form of Dunster's asymptotics gives

$$\rho_r = O_r(1), \quad \alpha_j = O_{r,j}(\lambda^{1/2-s_j}), \quad \beta_j = O_{r,j}(\lambda^{3/2-s_j}).$$

After the common strip factor is divided out, the Dini reference on a cell with left edge x is

$$\gg_{r,Y} \lambda^{-5/2} x^{2r-1}.$$

The transformed remainder is $O_r(1)$; it is dominated once

$$x \geq C_{r,Y} \lambda^{5/(2(2r-1))}.$$

The endpoint terms are smaller by the displayed powers of λ. Theorem 3.3 applies. Since $c = \lambda^2$, the stated exponent follows. Convergence is proposition 4.1. □

Corollary 4.4 (Verified compact core)**.** *Let*

$$H_0 = 3 \cdot 10^{12}.$$

For every fixed $0 < Y < 1/2$, Connes's published prolate transforms, and every endpoint-flat family in proposition 4.1 having the same limit, have only real simple zeros in

$$|\operatorname{Re} z| \le H_0, \qquad |\operatorname{Im} z| \le Y$$

for all sufficiently large cutoffs.

Proof. The published verification reaches beyond H_0, and the version-of-record statement explicitly records that every nontrivial zero through $3 \cdot 10^{12}$ is simple [7]. Thus all zeros of Ξ in the stated range are real and simple. Choose disjoint conjugation-invariant disks around the finitely many zeros and a zero-free compact complement. Local uniform convergence and Rouché's theorem give one zero in each disk and none in the complement. Conjugation symmetry makes each zero real. □

5 An explicit Gaussian–Hermite family

The prolate comparison shows that fixed endpoint order gives an arbitrarily small fixed power of c, but its proof uses fixed-mode asymptotics. We now use an exact limiting source and let the endpoint order grow.

Put

$$h(x) = x^2(2\pi x^2 - 3)e^{-\pi x^2}, \qquad g(x) = e^{-\pi x^2}.$$

The function h is a degree-zero/four Hermite combination, hence

$$\widehat{h} = h, \qquad h(0) = \int_{\mathbb{R}} h = 0.$$

For $\operatorname{Re} s > 1$, Gaussian integration gives

$$\begin{aligned} M_h(s) &:= \int_0^\infty h(x)x^{s-1}\,\mathrm{d}x \\ &= \frac{s(s-1)}{4}\pi^{-s/2-1}\Gamma(s/2), \end{aligned}$$

and therefore

$$\zeta(s)M_h(s) = \frac{\xi(s)}{2\pi}. \tag{5.1}$$

Poisson summation and self-duality continue this identity to the critical line and give

$$\int_{\mathbb{R}} \mathcal{E}(h)(e^y)\cos(zy)\,\mathrm{d}y = \frac{\Xi(z)}{2\pi}.$$

Let

$$r = \lfloor \log \lambda \rfloor, \qquad m = 2r + 2, \qquad w_\lambda(x) = \left(1 - \frac{x^2}{\lambda^2}\right)^m,$$

$$b_\lambda = \frac{\int_{-\lambda}^{\lambda} w_\lambda h}{\int_{-\lambda}^{\lambda} w_\lambda g}, \qquad q_\lambda = \mathbf{1}_{|x|\le\lambda} w_\lambda (h - b_\lambda g).$$

Theorem 5.1 (Logarithmic core)**.** *For every* $0 < Y < 1/2$*,*

$$\sup_{\substack{\operatorname{Re} z\in\mathbb{R}\\ |\operatorname{Im} z|\le Y}} \left| F_{\lambda,q_\lambda}(z) - \frac{\Xi(z)}{2\pi}\right| = O_Y((\log\lambda)\lambda^{-3/2+Y}).$$

There is K_Y *such that every zero in this strip with*

$$|\operatorname{Re} z| \ge K_Y \log\lambda$$

is real and simple for all sufficiently large λ*.*

Proof. The exact second moment is

$$\int_{\mathbb{R}} x^2 h(x)\,\mathrm{d}x = \frac{3}{2\pi^2}.$$

Taylor expansion of w_λ, with the outer Gaussian range estimated separately, gives

$$b_\lambda = -\frac{3m}{2\pi^2\lambda^2}\left(1 + O(m\lambda^{-2})\right),$$

and hence

$$q_\lambda(0) = -b_\lambda > 0, \qquad \|q_\lambda - h\|_1 = O(m\lambda^{-2}).$$

Lemma 2.1 and the Gaussian tails prove the convergence estimate. In angular normalization, the Dini coefficient satisfies

$$|A_\lambda| \ge c_0 r\lambda^{-5/2}. \tag{5.2}$$

Inside its support, write

$$q_\lambda(x) = P_{\lambda,r}(x)e^{-\pi x^2}, \qquad \deg P_{\lambda,r} \le 4r + 8.$$

For $R = A\sqrt{r}$, define

$$N_R(P) = \sum_j |a_j| R^j \quad \text{if} \quad P(x) = \sum_j a_j x^j.$$

Gaussian moments and Stirling give

$$\int_{\mathbb{R}} |P(x)| e^{-\pi x^2}\,\mathrm{d}x \le C N_R(P)$$

for degree $O(r)$. On polynomial factors,

$$\partial_x : P \mapsto P' - 2\pi x P, \qquad D : P \mapsto xP' - 2\pi x^2 P$$

cost $O(\sqrt{r})$ and $O(r)$, respectively, in N_R. Since $\mathcal{L} = -D(D+1)$, one application costs $O(r^2)$. Furthermore,

$$N_R(w_\lambda) = \left(1 + \frac{R^2}{\lambda^2}\right)^m \le 2.$$

It follows that, with $g_{j,\lambda} = \mathcal{L}^j q_\lambda$ and $s_j = 2(r-j)+2$,

$$\begin{aligned} M_r := \max\{\, &\|g_{r,\lambda}\|_1 + \left\|g''_{r,\lambda}\right\|_1, \\ &\left\|g_{j,\lambda}^{(s_j)}\right\|_1 + \left\|(xg_{j,\lambda})^{(s_j)}\right\|_1 : 0 \le j < r\} \le (Cr)^{2r+C}. \end{aligned} \tag{5.3}$$

The two endpoint zeros remaining after $\mathcal{L}^r$ give

$$\rho_r \le CM_r, \qquad |\alpha_j| \le CM_r\lambda^{1/2-s_j}, \qquad |\beta_j| \le CM_r\lambda^{3/2-s_j}.$$

On a normalized Dini cell with left edge x, the reference is at least

$$c_Y r\lambda^{-5/2}x^{2r-1}.$$

At $x \ge Kr$, the transformed-remainder ratio is

$$\ll_Y \lambda^{5/2}r^C(C/K)^{2r},$$

which tends to zero for sufficiently large fixed K. Dividing the endpoint terms by the reference gives

$$\frac{M_r}{r}\lambda^{1-2r+2j}x^{-2j}, \qquad \frac{M_r}{r}\lambda^{2-2r+2j}x^{-2j-1}.$$

Their maxima occur at $j = 0$ or $j = r-1$; the negative powers of λ and a further enlargement of K make their sum $o(1)$. Theorem 3.3 proves the zero assertion. □

Two variants delimit the mechanism.

Proposition 5.2 (Bounded tilts)**.** *Fix* $0 < a \le b$ *and* $\Theta < \infty$. *If*

$$a\log\lambda \le r_\lambda \le b\log\lambda, \qquad m_\lambda = 2r_\lambda + 2,$$

and w_λ *is multiplied by* $\exp(-\theta x^2/\lambda^2)$, $|\theta| \le \Theta$, *before the Gaussian zero-integral correction is made, then the convergence and* $O(\log\lambda)$ *real-tail conclusions of theorem 5.1 hold uniformly over this parameter family.*

Proof. Uniform Gaussian expansion gives

$$b_{\lambda,\theta} = -\frac{3(m_\lambda+\theta)}{2\pi^2\lambda^2} + O\left(\frac{(\log\lambda)^2}{\lambda^4}\right).$$

The interior Gaussian parameter remains in a compact positive interval, so (5.3) is uniform. Since $r_\lambda \ge a\log\lambda$, the same Dini comparison applies after choosing K in terms of a, b, Θ, Y. □

Proposition 5.3 (Balanced center)**.** *Let* $m = 2\lfloor\log\lambda\rfloor + 2$. *There is a unique* θ_λ *near* $-m$ *for which*

$$q_\lambda^{\mathrm{bal}}(x) = \mathbf{1}_{|x|\le\lambda}\left(1-\frac{x^2}{\lambda^2}\right)^m e^{-\theta_\lambda x^2/\lambda^2}h(x)$$

has zero integral. Moreover,

$$\theta_\lambda = -m - \frac{5m}{2\pi\lambda^2} + O(m\lambda^{-4}), \qquad q_\lambda^{\mathrm{bal}}(0) = 0,$$

$$\left\|q_\lambda^{\mathrm{bal}} - h\right\|_1 = O((\log\lambda)\lambda^{-4}),$$

and its finite transform converges to $\Xi/(2\pi)$ *over every whole closed substrip with error*

$$O_Y((\log\lambda)\lambda^{-7/2+Y}).$$

Proof. Use

$$\int x^2 h = \frac{3}{2\pi^2}, \qquad \int x^4 h = \frac{15}{2\pi^3},$$

and expand the combined profile in powers of λ^{-2}. The derivative of the integral with respect to θ is nonzero in the indicated window, so the implicit function theorem gives the unique solution and its expansion. The first deformation cancels, leaving the displayed L^1-error. Apply lemma 2.1. Since the source contains the factor h, its center is exactly zero. □

Remark 5.4. *The balanced-center family removes the universal Dini coefficient. It approximates a full Mellin transform containing the complete zeta divisor with superexponentially small compactification error, but this does not yield a real-tail theorem. It is a useful warning: a better uniform approximation can preserve a hypothetical off-axis zero more faithfully rather than exclude it.*

6 Shifted Euler heat

Let

$$D = x\partial_x, \qquad \mathcal{L} = -D(D+1), \qquad \mathcal{Z} = \mathcal{L} - \frac{1}{4} = -\left(D + \frac{1}{2}\right)^2.$$

On $L^2(\mathbb{R}_+, \mathrm{d}x)$, the logarithmic unitary map

$$(\mathcal{U}f)(y) = e^{y/2} f(e^y)$$

satisfies

$$\mathcal{U}\mathcal{Z}\mathcal{U}^{-1} = -\partial_y^2. \tag{6.1}$$

The ordinary Fourier transform changes $D + 1/2$ to its negative, so it commutes with $\mathcal{Z}$.

For $\alpha > 0$, put

$$H_\alpha = e^{-\alpha\mathcal{Z}} h, \qquad G_\alpha = e^{-\alpha/4} e^{-\alpha\mathcal{Z}} g.$$

Both are Fourier self-dual,

$$H_\alpha(0) = \int H_\alpha = 0, \qquad G_\alpha(0) = \int G_\alpha = 1,$$

and

$$M_{H_\alpha}(1/2 + iz) = e^{-\alpha z^2} M_h(1/2 + iz).$$

Theorem 6.1 (Polynomially flattened heat source)**.** *There is $\alpha_0 > 0$ such that, for each fixed $0 < \alpha < \alpha_0$, the polynomial endpoint construction*

$$q_{\lambda,\alpha} = \mathbf{1}_{|x|\le\lambda} \left(1 - \frac{x^2}{\lambda^2}\right)^{2r+2} (H_\alpha - b_{\lambda,\alpha} G_\alpha), \qquad r = \lfloor \log \lambda \rfloor,$$

with $b_{\lambda,\alpha}$ chosen to restore zero integral, satisfies

$$F_{\lambda, q_{\lambda,\alpha}} \longrightarrow \frac{\Xi(z)}{2\pi} e^{-\alpha z^2}$$

uniformly over every whole closed substrip. Every zero in $|\operatorname{Im} z| \le Y < 1/2$ with

$$|\operatorname{Re} z| \ge K_{\alpha,Y} \sqrt{\log \lambda}$$

is real and simple for all sufficiently large λ.

Proof. In logarithmic coordinates, $e^{-\alpha\mathcal{Z}}$ is convolution with the Gaussian of variance 2α. Thus

$$H_\alpha(x) = \mathbb{E}[e^{U/2}h(xe^U)].$$

Weighted convolution gives

$$\int e^{\sigma y}|\partial_y^n(\gamma_\alpha * V)(y)|\,\mathrm{d}y \le e^{\alpha\sigma^2}[C((n+1)/\alpha+\sigma^2)]^{n/2}\int e^{\sigma y}|V(y)|\,\mathrm{d}y. \tag{6.2}$$

The exact second moment becomes

$$\int x^2 H_\alpha(x)\,\mathrm{d}x = e^{25\alpha/4}\frac{3}{2\pi^2}.$$

Consequently

$$b_{\lambda,\alpha} = -\frac{3(2r+2)}{2\pi^2\lambda^2}e^{25\alpha/4}(1+O_\alpha(r\lambda^{-2})),$$

so

$$|A_{\lambda,\alpha}| \asymp_\alpha r\lambda^{-5/2}.$$

The source error is $O_\alpha(r\lambda^{-2})$, while the omitted lognormal tails are $\exp(-c_\alpha(\log\lambda)^2)$. Lemma 2.1 proves convergence.

Expand the endpoint polynomial. If $s_j = 2(r-j)+2$, weighted Young convolution, ordinary Hermite derivative bounds, and Stirling give

$$\begin{aligned} E_{j,\lambda,\alpha} &:= \left\|(\mathcal{L}^j q_{\lambda,\alpha})^{(s_j)}\right\|_1 + \left\|(x\mathcal{L}^j q_{\lambda,\alpha})^{(s_j)}\right\|_1 \\ &\le e^{C\alpha s_j^2}[C_\alpha(r+s_j^2)]^{j+C}[C(s_j+1)]^{s_j/2+C}. \end{aligned} \tag{6.3}$$

The term of endpoint-polynomial degree ℓ has additional logarithmic cost

$$-2\ell\log\lambda + C\alpha\ell^2 + C\ell\log(r+2).$$

For sufficiently small fixed α, this is at most $-c\ell\log\lambda$. Hence

$$\rho_r \le (C_\alpha r)^{r+C},$$

$$|\alpha_j| \le CE_{j,\lambda,\alpha}\lambda^{1/2-s_j}, \qquad |\beta_j| \le CE_{j,\lambda,\alpha}\lambda^{3/2-s_j}.$$

At $x \ge K\sqrt{r}$, the transformed-remainder ratio is

$$\ll_{\alpha,Y} \lambda^{5/2}r^C(C_\alpha/K^2)^r.$$

For endpoint terms, put $u = r-j$. If $u \le \sqrt{r}$, the $x^{-2(r-u)}$ factor dominates (6.3); if $u \ge \sqrt{r}$, the reserve $-2u\log\lambda$ dominates $C\alpha u^2 + O_\alpha(r\log r)$ after decreasing α_0. The sum of all endpoint ratios tends to zero. Apply theorem 3.3. ◻

7 Superheat and arbitrarily small powers

We now prove theorem 1.1. Fix $k \ge 1$ and $\alpha > 0$, and set

$$\mathcal{S}_{\alpha,k} = e^{-\alpha\mathcal{Z}^k}.$$

Under (6.1), this is the Fourier multiplier $e^{-\alpha\xi^{2k}}$. Let $K_{\alpha,k}$ denote its convolution kernel in the logarithmic variable.

Lemma 7.1 (Weighted superheat estimates)**.** *For every fixed real σ,*

$$\left\| e^{\sigma y} \partial_y^n K_{\alpha,k} \right\|_1 \le C_{\alpha,k,\sigma}^{n+1} (n+1)^{n/(2k)+C_{\alpha,k,\sigma}}. \tag{7.1}$$

More uniformly,

$$\begin{aligned} \log \left\| e^{\sigma y} \partial_y^n K_{\alpha,k} \right\|_1 \le{}& C_{\alpha,k}(1+|\sigma|)^{2k} \\ &+ \left(\frac{n}{2k} + C_{\alpha,k} \right) \log(n+2) + C_{\alpha,k} n \log(2+|\sigma|). \end{aligned} \tag{7.2}$$

If S is the superheat evolution of a polynomial times $e^{-\pi x^2}$, then, for fixed ordinary derivative order a, fixed polynomial weight M, and $0 \le j \le T$,

$$\begin{aligned} &\int_{|x| \ge e^T} (1+|x|)^M |\partial_x^a \mathcal{L}^j S(x)| \,\mathrm{d}x \\ &\le \exp\left(-c_{\alpha,k} T^{2k/(2k-1)} + C_{\alpha,k}(T+j+M+a+1)\log(T+j+M+a+2) \right). \end{aligned} \tag{7.3}$$

Proof. Contour translation in the inverse Fourier integral uses

$$\mathrm{Re}(\xi + i\sigma)^{2k} \ge c|\xi|^{2k} - C(1+|\sigma|)^{2k}.$$

For any function f,

$$\|f\|_1 \le C(\left\|\widehat{f}\right\|_2 + \left\|\partial_\xi^2 \widehat{f}\right\|_2).$$

Apply this after the contour shift to $(i\xi)^n e^{-\alpha \xi^{2k}}$. Gamma integration and Stirling yield (7.1); retaining the polynomial dependence on σ gives (7.2).

Weighted Young convolution transfers these estimates to the polynomial–Gaussian inputs. Markov's inequality with an extra logarithmic weight $\tau > 0$ gives an exponent

$$-\tau T + C\tau^{2k} + C(T+j+M+a+1)\log(T+j+M+a+2).$$

Choosing $\tau \asymp T^{1/(2k-1)}$ proves (7.3). □

Use the three Fourier-self-dual sources

$$h(x) = x^2(2\pi x^2 - 3)e^{-\pi x^2}, \quad g(x) = e^{-\pi x^2}, \quad r_0(x) = (1 - 2\pi x^2)e^{-\pi x^2}.$$

Let

$$c_{\alpha,k}^* = \exp\left(\alpha(-1/4)^k\right),$$

$$H = \mathcal{S}_{\alpha,k} h, \qquad G = c_{\alpha,k}^* \mathcal{S}_{\alpha,k} g, \qquad R = c_{\alpha,k}^* \mathcal{S}_{\alpha,k} r_0.$$

Evaluation at zero is an eigenfunctional of $\mathcal{Z}$ with eigenvalue $-1/4$, and Fourier self-duality exchanges center and integral. Therefore

$$H(0) = \int H = 0, \qquad G(0) = \int G = 1, \qquad R(0) = 1, \qquad \int R = 0. \tag{7.4}$$

Moreover,

$$M_H(1/2 + iz) = e^{-\alpha z^{2k}} M_h(1/2 + iz). \tag{7.5}$$

Choose a fixed Gevrey cutoff ϑ of order

$$s = 1 + \frac{1}{2k},$$

equal to one on $(-\infty, 1/2]$, flat and zero on $[1, \infty)$, and satisfying

$$\left\| \vartheta^{(n)} \right\|_\infty \le C^{n+1}(n+1)^{sn}.$$

With $L = \log \lambda$, set

$$\chi_\lambda(x) = \begin{cases} 1, & |x| \le 1, \\ \vartheta(\log |x|/L), & |x| > 1. \end{cases} \tag{7.6}$$

Then $\chi_\lambda = 1$ on $|x| \le \sqrt{\lambda}$, it is supported in $[-\lambda, \lambda]$, it is flat at both endpoints, and

$$\|D^n \chi_\lambda\|_\infty \le C^{n+1}(n+1)^{sn} L^{-n}. \tag{7.7}$$

Put

$$a_\lambda = \lambda^{-2}, \qquad b_\lambda = \frac{\int \chi_\lambda(H + a_\lambda R)}{\int \chi_\lambda G},$$

$$q_{\lambda,\alpha,k} = \chi_\lambda(H + a_\lambda R - b_\lambda G). \tag{7.8}$$

Lemma 7.2 (Compactification bounds)**.** *The source in (7.8) is real, even, compact, endpoint-flat, and has zero integral. As $\lambda \to \infty$,*

$$b_\lambda = O\left(e^{-c(\log \lambda)^{2k/(2k-1)}}\right), \qquad q_{\lambda,\alpha,k}(0) = \lambda^{-2}(1 + o(1)), \tag{7.9}$$

$$\|q_{\lambda,\alpha,k} - H\|_1 = O(\lambda^{-2}). \tag{7.10}$$

If $r = \lfloor \log \lambda \rfloor$, then

$$\|\mathcal{L}^r q_{\lambda,\alpha,k}\|_1 + \|(\mathcal{L}^r q_{\lambda,\alpha,k})''\|_1 \le (C_{\alpha,k} r)^{r/k + C_{\alpha,k}}. \tag{7.11}$$

Proof. Only the outer x-tail is cut off; χ_λ is identically one near zero. Equations (7.3) and (7.4) show that the numerator defining b_λ is super-stretched small, while its denominator tends to one. This gives (7.9). Equation (7.10) follows from $a_\lambda = \lambda^{-2}$, bounded L^1-norms of R, G, and the same tail estimate.

For (7.11), expand $\mathcal{L}^r$ into at most C^r products of $2r$ Euler derivatives and lower-order terms. If n derivatives land on the cutoff, (7.7) costs

$$\exp\left(\frac{n}{2k} \log r + O_{\alpha,k}(r)\right).$$

The remaining $2r - n$ logarithmic derivatives land on a superheat source. Equation (7.2), with the finitely many weights introduced by ordinary derivatives, costs

$$\exp\left(\frac{2r - n}{2k} \log r + O_{\alpha,k}(r)\right).$$

Every Leibniz term therefore has the same leading exponent $(r/k) \log r + O_{\alpha,k}(r)$. The binomial and lower-order combinatorics are absorbed into the $O(r)$ term. The two ordinary derivatives are handled by the fixed shifted operators arising under $\mathcal{U}$. Summation proves (7.11). □

Lemma 7.3 (Alias bounds)**.** *Let* $p_c(v) = \lambda^{1/2} q_{\lambda,\alpha,k}(\lambda v)$*,* $c = \lambda^2$*, and let* K_c *be its arithmetic kernel. For* $r = \lfloor \log \lambda \rfloor$*, the Dini coefficient and comparison data satisfy*

$$A_c = -\frac{1}{2}\lambda^{-5/2}(1 + o(1)), \tag{7.12}$$

$$\rho_r := \int |B^r K_c| \le (C_{\alpha,k} r)^{r/k + C_{\alpha,k}}, \tag{7.13}$$

and, uniformly for $0 \le j < r$*,*

$$|\alpha_j| + |\beta_j| \le \exp\left(-c_{\alpha,k}(\log \lambda)^{2k/(2k-1)} + C_{\alpha,k} r \log r\right). \tag{7.14}$$

Proof. Equation (7.12) is proposition 2.2 together with (7.9). The arithmetic–Euler intertwining is

$$B^j \mathcal{E}(f) = \mathcal{E}(\mathcal{L}^j f).$$

Endpoint flatness leaves arbitrarily many spare zeros, so the zero extension has no boundary distributions. Two ordinary integrations by parts and (7.11) give (7.13).

For the endpoint aliases, split the Fourier transform into the uncut superheat source and the cutoff error. Fourier commutation puts the uncut sampled values at $n\lambda$ in the same class controlled by (7.3). The cutoff error is supported where $|x| \ge \sqrt{\lambda}$; repeated integration by parts makes its sampled series absolutely summable, and (7.3) supplies the same leading negative exponent. Applying at most $2r + 1$ Euler or ordinary derivatives contributes $O_{\alpha,k}(r \log r)$. Summation over $n \ge 1$ is absorbed by decreasing the leading constant, proving (7.14). □

Proof of theorem 1.1. The source properties follow from (7.8). Equations (5.1), (7.5), (7.10), and lemma 2.1 give

$$\sup_{\substack{\operatorname{Re} z \in \mathbb{R} \\ |\operatorname{Im} z| \le Y}} \left| F_{\lambda, q_{\lambda,\alpha,k}}(z) - \frac{\Xi(z)}{2\pi} e^{-\alpha z^{2k}} \right| = O_{\alpha,k,Y}(\lambda^{-3/2+Y}).$$

The omitted logarithmic tails of the uncut transform are super-stretched small by (7.3) and Fourier self-duality.

On a normalized Dini cell with left edge x, the reference is at least

$$c_{\alpha,k,Y} \lambda^{-5/2} x^{2r-1}.$$

By (7.13), the transformed-remainder ratio is at most

$$\lambda^{5/2} (C_{\alpha,k} r)^{r/k + C_{\alpha,k}} x^{-(2r-1)}.$$

Set

$$x \ge K_{\alpha,k,Y} r^{1/(2k)}.$$

The $r \log r/k$ terms cancel. A sufficiently large fixed K makes the remaining exponential in r dominate $\lambda^{5/2}$, since $r = \log \lambda + O(1)$. Equation (7.14) is smaller than every fixed power of λ^{-1}, so all endpoint-defect terms are negligible on this scale. Theorem 3.3 gives one real simple zero in every complete remote cell. Reflection handles the negative real direction. □

Proof of corollary 1.2. Choose a fixed integer $k > 1/(2\varepsilon)$. Since $\log \lambda = \frac{1}{2} \log c$, the exponent $1/(2k)$ is smaller than ε. □

8 Limitations and zero-divisor fidelity

The factor $e^{-\alpha z^{2k}}$ is entire and zero-free, so the limiting function in theorem 1.1 has exactly the zero divisor of Ξ. This fidelity does not turn the theorem into an RH criterion. Every fixed hypothetical off-axis zero of Ξ eventually lies inside the expanding core. Whole-strip absolute convergence does not close the gap: the gamma factor makes Ξ exponentially small along the real direction, while the available approximation error is only a negative power of the cutoff.

The conclusion also depends on the arithmetic source, not merely on self-duality and regularity. For example, replace h by

$$\widetilde{h} = (1 + 16\mathcal{Z}^2)h.$$

All source estimates above remain valid, but the limiting transform is

$$\frac{\Xi(z)}{2\pi}(1 + 16z^4)e^{-\alpha z^{2k}},$$

which has four explicit off-axis zeros with imaginary part $1/(2\sqrt{2}) < 1/2$. The same real-tail theorem therefore coexists with nonreal zeros in the expanding core.

Three restrictions should be kept explicit.

(i) The order k is fixed. No constants are uniform as $k \to \infty$.

(ii) The theorem controls complete remote Dini cells. It gives no hyperbolicity statement in the central transition region.

(iii) Numerical zero frontiers are not used in any proof. All asymptotic conclusions follow from the analytic inequalities stated above.

9 Data and code availability

The results are analytic and use no experimental data. A reproducibility archive accompanies the manuscript. It contains the exact source, source-pin manifest, deterministic build instructions, and checks of the normalizations, Mellin identities, Dini characteristic equation, theorem markers, bibliography coverage, and archive integrity. No numerical frontier is part of the proof.

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