THE DE BRUIJN–NEWMAN KERNEL HAS EXACT PÓLYA-FREQUENCY ORDER FOUR

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Abstract. Let

$$\Phi(t)=\sum_{n\geq 1}\left(2\pi^2 n^4 \mathrm{e}^{9t}-3\pi n^2 \mathrm{e}^{5t}\right)\mathrm{e}^{-\pi n^2 \mathrm{e}^{4t}}, \qquad t\geq 0,$$

and let $K(t)=\Phi(|t|)$ be its smooth even extension. We prove that K is strictly Pólya-frequency of order four: for every $1\leq r\leq 4$ and every two strictly increasing real packets $x_1<\cdots<x_r$, $y_1<\cdots<y_r$,

$$\det[K(x_i-y_j)]_{i,j=1}^{r}>0.$$

The proof reduces all ordered translation minors to the one-variable confluent flag

$$H_m(t)=(-1)^{m(m-1)/2}\det[K^{(i+j)}(t)]_{i,j=0}^{m-1}$$

by two extended-complete-Chebyshev collocation steps. We certify $H_m(t)>0$ for $m\leq 4$ using a 65-cell interval-arithmetic cover of $0\leq t\leq 13/32$, followed by an analytic first-theta-shell dominance argument on the infinite tail. Collision limits are controlled by the same flag and explicit Vandermonde normalization.

Michałowski recently certified a negative order-five Toeplitz minor and left global PF_4 as an open problem. We independently reconstruct his order-five witness with Arb. Consequently the exact Pólya-frequency order of the de Bruijn–Newman kernel is four.

1. Introduction

The classical de Bruijn–Newman kernel is the positive even theta kernel whose cosine transform gives the Riemann Ξ-function, up to a harmless normalization. In the normalization used here,

$$\Phi(t)=\sum_{n\geq 1}\left(2\pi^2 n^4 \mathrm{e}^{9t}-3\pi n^2 \mathrm{e}^{5t}\right)\mathrm{e}^{-\pi n^2 \mathrm{e}^{4t}}, \qquad t\geq 0, \tag{1}$$

and

$$K(t)=\Phi(|t|).$$

Date: July 22, 2026.

2020 *Mathematics Subject Classification.* Primary 15B48; Secondary 11M26, 41A50, 65G30.

Key words and phrases. de Bruijn–Newman kernel, Pólya-frequency function, total positivity, extended complete Chebyshev system, interval arithmetic, theta function.

To make the smooth extension explicit, put

$$\vartheta(y) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi n^2 y}.$$

Then

$$\Phi(t) = \mathrm{e}^{9t}\vartheta''(\mathrm{e}^{4t}) + \frac{3}{2}\mathrm{e}^{5t}\vartheta'(\mathrm{e}^{4t}).$$

The Jacobi relation

$$\vartheta(y) = y^{-1/2}\vartheta(1/y)$$

and two differentiations show directly that $\Phi(-t) = \Phi(t)$. The theta series is real analytic for $y > 0$, so the same formula defines a real-analytic even function through $t = 0$; it agrees with $K(t) = \Phi(|t|)$.

This kernel sits naturally between two classical subjects: the heat-flow approach to the Riemann hypothesis [1, 6, 8, 9] and the theory of totally positive translation kernels [10, 3, 7].

A function $f : \mathbb{R} \to \mathbb{R}$ is Pólya-frequency of order r, written $f \in \mathrm{PF}_r$, if

$$\det[f(x_i - y_j)]_{i,j=1}^{m} \geq 0$$

for every $1 \leq m \leq r$ and all increasing real packets $x_1 \leq \cdots \leq x_m$, $y_1 \leq \cdots \leq y_m$. It is strictly PF_r when the determinant is positive whenever both packets are strictly increasing.

Michałowski [5] recently proved by certified interval arithmetic that the kernel (1) is not PF_5. The same paper verified positive determinants of orders two through four at the displayed order-five counterexample, but emphasized that those signs do not imply global PF_4; global PF_4 was stated as its first open problem.

We resolve that problem.

Theorem 1.1. *The de Bruijn–Newman kernel K is strictly* PF_4. *More precisely, for every $1 \leq m \leq 4$ and all strictly increasing real packets*

$$x_1 < \cdots < x_m, \qquad y_1 < \cdots < y_m,$$

one has

$$\det[K(x_i - y_j)]_{i,j=1}^{m} > 0.$$

For weakly increasing packets the determinant is nonnegative.

At order five,

$$\det\left[K\left(\frac{1}{100} + \frac{i-j}{20}\right)\right]_{i,j=0}^{4} < 0.$$

Consequently K has exact Pólya-frequency order four.

The conceptual step is a complete confluent-flag criterion. Define

$$H_m(t) = (-1)^{m(m-1)/2} \det[K^{(i+j)}(t)]_{i,j=0}^{m-1}. \tag{2}$$

Positivity of $H_1, \ldots, H_r$ on the line implies strict PF_r, not merely positivity in a small-spacing or fully confluent limit. One application of the extended-complete-Chebyshev collocation theorem turns the flag into derivative collocation determinants; a second application turns those determinants into arbitrary translation minors. Thus a $2m$-variable configuration problem collapses to m one-variable inequalities.

For the kernel (1), the one-variable inequalities admit a short computer-assisted proof. On a compact interval we use exact theta-shell derivative polynomials, rigorous omitted-shell bounds, and Taylor models on 65 rational cells. Beyond the compact interval, the first theta shell has explicitly factored Hankel determinants. The sum of all higher shells is then bounded by a perturbation that decreases monotonically while the first-shell determinant increases.

The paper is organized as follows. Section 2 gives the two-step Chebyshev reduction and the collision formulas. Section 3 derives the exact theta-shell differential calculus. Section 4 describes the compact certificate, and Section 5 proves the analytic tail. Section 6 completes the total-positivity argument and reconstructs the order-five witness. Section 7 specifies the executable proof artifact.

Remark 1.2. *The theorem concerns the fixed zero-heat kernel. It is not an all-order total-positivity statement, a heat-neighborhood statement, or a criterion for the Riemann hypothesis. Indeed, order five already fails.*

2. From the complete confluent flag to all minors

We use the following standard collocation theorem for extended complete Chebyshev systems; see Karlin and Studden [4, Chapter I] or Karlin [3, Chapter XI].

Lemma 2.1 (ECT collocation). *Let I be an interval and let $f_0, \dots, f_{r-1} \in C^{r-1}(I)$. Suppose every initial Wronskian is strictly positive:*

$$W(f_0, \dots, f_{m-1})(t) > 0, \qquad 1 \le m \le r, \quad t \in I.$$

Then for every $1 \le m \le r$ and every $t_1 < \cdots < t_m$ in I,

$$\det[f_{i-1}(t_j)]_{i,j=1}^{m} > 0.$$

The corresponding confluent collocation determinants are nonnegative on the weakly ordered closure and positive after division by the within-cluster Vandermonde factors.

Proposition 2.2 (Two-step translation criterion). *Let $F \in C^{2r-2}(\mathbb{R})$ be positive. If*

$$(-1)^{m(m-1)/2} \det[F^{(i+j)}(t)]_{i,j=0}^{m-1} > 0$$

for every $1 \le m \le r$ and every $t \in \mathbb{R}$, then F is strictly PF_r*.*

Proof. Fix $x \in \mathbb{R}$, and for $c = 0, \dots, r-1$ put

$$g_c(y) = F^{(c)}(x - y).$$

For $1 \le m \le r$, the initial Wronskian in y is

$$\begin{aligned} W_y(g_0, \dots, g_{m-1})(y) &= \det[(-1)^i F^{(i+j)}(x-y)]_{i,j=0}^{m-1} \\ &= (-1)^{m(m-1)/2} \det[F^{(i+j)}(x-y)]_{i,j=0}^{m-1} > 0. \end{aligned}$$

Lemma 2.1 therefore gives, for every ordered $y_1 < \cdots < y_m$,

$$\det[F^{(i-1)}(x-y_j)]_{i,j=1}^{m} > 0. \tag{3}$$

Now fix the ordered y-packet and define

$$h_j(x) = F(x-y_j), \qquad 1 \le j \le m.$$

For each $k \le m$, the initial Wronskian

$$W_x(h_1,\dots,h_k)(x) = \det[F^{(i-1)}(x-y_j)]_{i,j=1}^{k}$$

is positive by (3). A second application of Lemma 2.1, now in the x-variable, yields

$$\det[F(x_i-y_j)]_{i,j=1}^{m} > 0$$

for every $x_1 < \cdots < x_m$. □

The fully confluent normalization will also be useful. Write

$$\Delta(z_1,\dots,z_m) = \prod_{1\le i<j\le m}(z_j-z_i).$$

Lemma 2.3 (Fully confluent limit)**.** *If $F \in C^{2m-2}$, then*

$$\lim_{\substack{x_i\to x\\ y_j\to y}} \frac{\det[F(x_i-y_j)]_{i,j=1}^{m}}{\Delta(x_1,\dots,x_m)\Delta(y_1,\dots,y_m)} = \frac{(-1)^{m(m-1)/2}\det[F^{(i+j)}(x-y)]_{i,j=0}^{m-1}}{\prod_{j=0}^{m-1}(j!)^2}.$$

Proof. Expand first in the x-variables and then in the y-variables. The standard confluent Vandermonde coefficient is

$$\frac{\det[\partial_x^i\partial_y^j F(x-y)]_{i,j=0}^{m-1}}{\prod_{j=0}^{m-1}(j!)^2}.$$

Since $\partial_x^i\partial_y^j F(x-y) = (-1)^j F^{(i+j)}(x-y)$, factoring $(-1)^j$ from column j gives the stated sign. □

Thus Theorem 1.1 reduces to proving

$$H_m(t) > 0, \qquad t\in\mathbb{R}, \quad 1 \le m \le 4. \tag{4}$$

Because K is even, each H_m is even: at $-t$, row i and column j contribute signs $(-1)^i$ and $(-1)^j$, whose products cancel in the determinant. It is therefore enough to establish (4) on $[0,\infty)$.

3. Exact theta-shell derivatives

For $t \ge 0$, set

$$x = \pi e^{4t}, \qquad x_n = n^2 x.$$

The n-th shell in (1) becomes

$$\left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right) e^{-x_n} = e^t x_n(2x_n-3)e^{-x_n}.$$

Define integer polynomials p_q by

$$p_0(z) = z(2z-3), \qquad p_{q+1}(z) = (1-4z)p_q(z) + 4zp_q'(z). \tag{5}$$

Lemma 3.1 (Shell differential calculus)**.** *For every integer $q \geq 0$ and every $t \geq 0$,*

$$K^{(q)}(t) = \mathrm{e}^{t} \sum_{n \geq 1} \mathrm{e}^{-x_n} p_q(x_n). \tag{6}$$

Moreover, $p_q \in \mathbb{Z}[z]$ has degree $q+2$.

Proof. For a polynomial p,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{t}\mathrm{e}^{-x_n} p(x_n)\right) = \mathrm{e}^{t}\mathrm{e}^{-x_n} \left((1-4x_n)p(x_n) + 4x_n p'(x_n)\right),$$

because $x_n' = 4x_n$. Starting from $p_0(z) = z(2z-3)$ gives (5) and (6) by induction. The recurrence raises the degree by one, and its leading coefficient does not vanish. □

Every summand of $K(t)$ is positive:

$$x_n(2x_n - 3) > 0 \qquad \text{because} \qquad x_n \geq \pi > \frac{3}{2}.$$

Hence

$$H_1(t) = K(t) > 0. \tag{7}$$

The remaining work is to prove positivity of H_2, H_3, H_4.

4. The compact interval certificate

We certify the flag on

$$0 \leq t \leq T, \qquad T = \frac{13}{32},$$

using exact rational cells

$$I_j = \left[\frac{j}{160}, \frac{j+1}{160}\right], \qquad 0 \leq j < 65. \tag{8}$$

Each cell has center $c_j = (2j+1)/320$ and radius $\rho = 1/320$.

4.1. **A uniform shell-tail bound.** Write

$$p_q(z) = \sum_d c_{q,d} z^d.$$

The verifier evaluates the first $N = 20$ shells directly with Arb. The omitted shells are bounded uniformly on a slightly larger parameter range

$$t \geq L, \qquad L = -\frac{1}{1000}.$$

The negative floor absorbs outward rounding at the left endpoint of the first cell.

For fixed n, d, consider

$$M_{n,d}(t) = \mathrm{e}^{t}(\pi n^2 \mathrm{e}^{4t})^d \mathrm{e}^{-\pi n^2 \mathrm{e}^{4t}}.$$

Its logarithmic derivative is

$$\frac{M'_{n,d}(t)}{M_{n,d}(t)} = 1 + 4d - 4\pi n^2 \mathrm{e}^{4t}.$$

For $n \geq 21$, $q \leq 15$, $d \leq q+2$, and $t \geq L$, this quantity is negative. Thus $M_{n,d}(t) \leq M_{n,d}(L)$.

At $t = L$, the ratio of consecutive n-terms is

$$q_{n,d} = \left(\frac{n+1}{n}\right)^{2d} \exp\left(-\pi(2n+1)\mathrm{e}^{4L}\right).$$

Both factors decrease with n, so

$$\sum_{n \geq 21} M_{n,d}(t) \leq \frac{M_{21,d}(L)}{1 - q_{21,d}}.$$

Summing against $|c_{q,d}|$ gives a rigorous enclosure of the complete omitted shell tail for every derivative through order 15.

4.2. **Taylor models for the flag.** Let

$$\epsilon_m = (-1)^{m(m-1)/2}.$$

Differentiating the determinant by the multinomial Leibniz rule gives the exact formula

$$\begin{aligned} H_m^{(s)}(t) = \epsilon_m \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \sum_{\substack{\alpha_0 + \cdots + \alpha_{m-1} = s \\ \alpha_i \geq 0}} \binom{s}{\alpha_0, \ldots, \alpha_{m-1}} \\ \times \prod_{i=0}^{m-1} K^{(i+\sigma(i)+\alpha_i)}(t). \end{aligned} \tag{9}$$

For $m \leq 4$ and $s \leq 9$, the largest required kernel derivative is 15, exactly the range covered above.

At the center c_j of a cell, the verifier computes $H_m^{(s)}(c_j)$ for $0 \leq s \leq 8$. On the full cell it uses (9), with absolute values and interval kernel jets, to bound $H_m^{(9)}$. Taylor's theorem gives

$$\begin{aligned} H_m(t) \geq \inf H_m(c_j) - \sum_{s=1}^{8} \frac{\sup|H_m^{(s)}(c_j)|}{s!} \rho^s \\ - \frac{\sup_{u \in I_j} |H_m^{(9)}(u)|}{9!} \rho^9, \qquad t \in I_j. \end{aligned} \tag{10}$$

Every operation is outward-rounded. The center value is also evaluated independently as an Arb matrix determinant and required to overlap the Leibniz expression.

Proposition 4.1 (Compact flag). *For $m = 2, 3, 4$,*

$$H_m(t) > 0, \qquad 0 \le t \le \frac{13}{32}.$$

The weakest certified lower bounds are as follows.

m	*cell index*	*lower bound on the cell*
2	64	$1.4737407298724188 \cdot 10^{-6}$
3	64	$1.2160040433993911 \cdot 10^{-5}$
4	64	$4.4937173449266680 \cdot 10^{-2}$

Proof. There are 65 cells and three nontrivial flag orders, hence 195 Taylor inequalities of the form (10). The pinned verifier evaluates all 195 and fails unless every lower endpoint is strictly positive. The displayed values are outward-rounded below the weakest certified lower endpoints. All three minima occur on the final cell $[2/5, 13/32]$. □

5. The analytic infinite tail

For $t \ge T = 13/32$, write

$$x = \pi e^{4t}, \qquad x \ge x_0 := \pi e^{13/8} = 15.9543239\ldots.$$

Factor the first theta shell from (6):

$$K^{(q)}(t) = e^t e^{-x} \left(p_q(x) + R_q(x)\right), \tag{11}$$

where

$$R_q(x) = \sum_{n\ge 2} e^{-(n^2-1)x} p_q(n^2 x). \tag{12}$$

The common factor $e^t e^{-x}$ contributes the positive scalar $(e^t e^{-x})^m$ to an order-m determinant, so only the matrix $[p_{i+j}(x) + R_{i+j}(x)]$ matters for signs.

5.1. **Exact first-shell determinants.** The recurrence (5) gives exact symbolic factorizations:

$$-\det[p_{i+j}(x)]_{i,j=0}^{1} = 16x^3 Q_2(x), \tag{13}$$

$$-\det[p_{i+j}(x)]_{i,j=0}^{2} = 8192x^6 Q_3(x), \tag{14}$$

$$\det[p_{i+j}(x)]_{i,j=0}^{3} = 201326592x^{10} Q_4(x), \tag{15}$$

where

$$Q_2(x) = 4x^2 - 12x + 15 = (2x-3)^2 + 6,$$
$$Q_3(x) = 8x^3 - 36x^2 + 90x - 105,$$
$$Q_4(x) = 16x^4 - 96x^3 + 360x^2 - 840x + 945.$$

The orientations in (13)–(15) are exactly those in (2). Moreover,

$$Q_3'(x) = 6Q_2(x), \qquad Q_4'(x) = 8Q_3(x),$$

and

$$Q_3(3) = 57, \qquad Q_4(3) = 369.$$

Thus all three oriented first-shell determinants are positive and increasing for $x \geq x_0 > 3$.

5.2. **All higher shells as a decreasing perturbation.** Let

$$d_q = q + 2, \qquad A_q = \sum_d |c_{q,d}|, \qquad B_q(x) = A_q x^{d_q}.$$

Since $n^2 x \geq 1$,

$$|p_q(n^2 x)| \leq A_q (n^2 x)^{d_q}.$$

The first term in (12), at $n = 2$, is bounded by

$$B_q(x) 4^{d_q} \mathrm{e}^{-3x}.$$

For consecutive $n \geq 2$, the ratio is at most

$$\left(\frac{3}{2}\right)^{2d_q} \mathrm{e}^{-5x}.$$

Hence

$$|R_q(x)| \leq B_q(x) \delta_q(x), \qquad \delta_q(x) = \frac{4^{d_q} \mathrm{e}^{-3x}}{1 - (3/2)^{2d_q} \mathrm{e}^{-5x}}. \tag{16}$$

For $m = 2, 3, 4$, determinant multilinearity and the Leibniz formula give

$$\begin{aligned} &|\det[p_{i+j}(x) + R_{i+j}(x)]_{i,j=0}^{m-1} - \det[p_{i+j}(x)]_{i,j=0}^{m-1}| \\ &\quad \leq E_m(x), \end{aligned} \tag{17}$$

where

$$E_m(x) = \sum_{\sigma \in S_m} \left[\prod_{i=0}^{m-1} B_{i+\sigma(i)}(x)\big(1 + \delta_{i+\sigma(i)}(x)\big) - \prod_{i=0}^{m-1} B_{i+\sigma(i)}(x) \right]. \tag{18}$$

After expanding the brackets, every nonzero term of E_m has the form

$$C x^{m(m+1)} \mathrm{e}^{-3sx} \prod_{\nu=1}^{s} \left(1 - a_\nu \mathrm{e}^{-5x}\right)^{-1}, \qquad C > 0, \quad s \geq 1. \tag{19}$$

The denominator factors decrease with x. The logarithmic derivative of the first factor is

$$\frac{m(m+1)}{x} - 3s < 0$$

for $x \geq x_0$, because

$$x_0 > \frac{m(m+1)}{3} \qquad (m \leq 4).$$

Thus $E_m(x)$ decreases on the entire tail.

Proposition 5.1 (Tail flag)**.** *For $m = 2, 3, 4$,*

$$H_m(t) > 0, \qquad t \geq \frac{13}{32}.$$

Proof. The oriented first-shell determinants increase with x, and the error majorants $E_m(x)$ decrease. It is enough to compare them at $x_0 = \pi e^{13/8}$. Arb gives:

m	quantity	certified bound
2	first shell, lower	$5.4691224458717487 \cdot 10^{7}$
	$E_2(x_0)$, upper	$3.3923702659133560 \cdot 10^{-8}$
	reserve, lower	$5.4691224458717453 \cdot 10^{7}$
3	first shell, lower	$3.3310100708125457 \cdot 10^{15}$
	$E_3(x_0)$, upper	$2.6719179125881913 \cdot 10^{6}$
	reserve, lower	$3.3310100681406278 \cdot 10^{15}$
4	first shell, lower	$1.5617298407963874 \cdot 10^{26}$
	$E_4(x_0)$, upper	$2.3450366119067967 \cdot 10^{25}$
	reserve, lower	$1.3272261796057078 \cdot 10^{26}$

Every reserve is positive. Equations (11)–(17) then prove the stated tail positivity. □

6. Completion of the proof and the first failure

Proof of Theorem 1.1. Equation (7), Proposition 4.1, and Proposition 5.1 prove

$$H_m(t) > 0, \qquad t \geq 0, \quad 1 \leq m \leq 4.$$

Evenness extends this to every $t \in \mathbb{R}$. Proposition 2.2 now gives strict positivity of every ordered translation minor through order four.

If a row or column packet has an exact collision, the determinant has repeated rows or columns and is zero. Continuity gives nonnegativity on the weakly ordered closure. Lemma 2.3 gives the explicit fully confluent normalization

$$\lim_{\substack{x_i \to x \\ y_j \to y}} \frac{\det[K(x_i - y_j)]}{\Delta(x_1, \ldots, x_m)\Delta(y_1, \ldots, y_m)} = \frac{H_m(x-y)}{\prod_{j=0}^{m-1}(j!)^2} > 0.$$

Partial collisions are the corresponding generalized ECT collocation limits and are likewise positive after division by their within-cluster Vandermonde factors.

Finally, put

$$D_5 = \det\left[K\left(\frac{1}{100} + \frac{i-j}{20}\right)\right]_{i,j=0}^{4}.$$

The pinned Arb evaluator gives

$$-1.847236073442659 \cdot 10^{-9} < D_5 < -1.847236073442658 \cdot 10^{-9}.$$

Thus $K \notin \mathrm{PF}_5$, while $K \in \mathrm{PF}_4$. □

For comparison, the same equally spaced Toeplitz geometry gives positive lower orders:

$$D_2 = 0.0340640406226313692301\ldots > 0,$$
$$D_3 = 0.0006976970642643794806\ldots > 0,$$
$$D_4 = 3.8274713675688884527\ldots \cdot 10^{-6} > 0,$$
$$D_5 = -1.8472360734426587333\ldots \cdot 10^{-9} < 0.$$

This reconstructs the certified order-five witness of [5] independently of the derivative pipeline that was withdrawn between versions one and two of that paper.

Remark 6.1 (No hidden compactness assumption)**.** *The ECT conclusion applies directly to every finite ordered configuration on the whole real line. No bound on offsets or gaps is introduced. Consequently noncompact configuration sequences require no separate uniform chamber argument: each member is already covered, and any finite limit of their nonnegative determinants remains nonnegative.*

7. The executable certificate

The public source package contains the manuscript, the complete verifier, a frozen dependency lock, and accepted output. The verifier is fail-closed under optimized Python and performs the following checks.

(1) It constructs the integer polynomials p_q through $q = 15$ and checks recurrence (5) symbolically.
(2) It checks the exact factorizations (13)–(15) and the derivative identities for Q_2, Q_3, Q_4.
(3) It checks the determinant orientations used by the two ECT steps and the fully confluent formula.
(4) It evaluates all 195 compact-cell inequalities, with 20 direct theta shells and a rigorous complete omitted-shell bound.
(5) It verifies the seam values and positive reserves in Proposition 5.1.
(6) It independently reconstructs the order-two through order-five Toeplitz determinants.

The computation uses Python 3.14.6, python-flint 0.9.0, FLINT 3.6.0, and SymPy 1.14.0. Arb precision is set to 120 decimal digits. All loops are fixed and bounded: 65 cells, three nontrivial flag orders, 20 direct shells, Taylor degree eight, and kernel jets through order 15.

From the extracted source archive, run

```
uv run --frozen python verify.py
```

to reproduce the certificate. The accepted run prints a JSON object with `"status": "certified"` and exits with status zero.

The numerical proof is not a grid-sign argument. Every compact cell is covered by the Taylor remainder inequality (10), and the infinite tail is covered by the monotonic comparison in Section 5. Thus the certificate covers the continuum $[0, \infty)$, not a sampled subset.

8. Context and boundary

Schoenberg's theory characterizes all-order Pólya-frequency functions through their bilateral Laplace transforms [10, 3]. The kernel (1) lies strictly below that all-order class because its order-five minor is negative. The present result instead determines its exact finite order.

Finite-order total positivity has a different flavor from the all-order classification. In general, positivity of a few confluent determinants need not be mistaken for positivity of all nonconfluent minors. What makes the reduction work here is the *complete initial flag* $H_1, \dots, H_r$: it supplies an ECT system once in the column variable and a second ECT system in the row variable. This is why the proof covers arbitrary packets rather than only small-spacing Toeplitz configurations.

The arithmetic content enters in the proof of the flag itself. A generic positive even Schwartz kernel need not satisfy even the order-two inequality. For example,

$$G(t) = \mathrm{e}^{-(t-1)^2} + \mathrm{e}^{-(t+1)^2}$$

is positive, even, and rapidly decreasing, but

$$G'(0)^2 - G(0)G''(0) = -8\mathrm{e}^{-2} < 0.$$

The compact certificate and tail factorization use the literal integer-square theta shells of (1); positivity and evenness alone do not explain the result.

The exact order-four boundary is therefore sharp in both directions: every minor through order four is positive at strict configurations, while a regular equally spaced configuration already fails at order five.

Acknowledgments

The author thanks Wojciech Michałowski for isolating the finite-order question and for the certified order-five witness in [5]. The Arb library made the compact continuum certificate pleasantly small.

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