

The generalized Laguerre inequalities and functions in the Laguerre–Pólya class

Research Article

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Received 6 June 2012; accepted 14 October 2012

Abstract: The principal goal of this paper is to show that the various sufficient conditions for a real entire function, $\varphi(x)$, to belong to the Laguerre–Pólya class (Definition 1.1), expressed in terms of Laguerre-type inequalities, do not require the *a priori* assumptions about the order and type of $\varphi(x)$. The proof of the main theorem (Theorem 2.3) involving the generalized real Laguerre inequalities, is based on a beautiful geometric result, the Borel–Carathéodory Inequality (Theorem 2.1), and on a deep theorem of Lindelöf (Theorem 2.2). In case of the complex Laguerre inequalities (Theorem 3.2), the proof is sketched for it requires a slightly more delicate analysis. Section 3 concludes with some other cognate results, an open problem and a conjecture which is based on Cardon's recent, ingenious extension of the Laguerre-type inequalities.

MSC: 30D15, 30D35

Keywords: Laguerre–Pólya class • Generalized Laguerre-type inequalities
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1. Introduction

The main leitmotif of this note pertains to certain inequalities used in the characterization of functions in the Laguerre–Pólya class. In the present investigation, it will be convenient to adopt the following, classical definition of this family of functions.

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Definition 1.1.

A real entire function $\varphi(x) = \sum_{k=0}^{\infty} (\gamma_k/k!)x^k$ is said to be in the *Laguerre–Pólya class*, written $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, if $\varphi(x)$ can be expressed in the form

$$\varphi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k}, \quad 0 \leq \omega \leq \infty,$$

where $c, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. If $\omega = 0$, then, by convention, the product is defined to be 1.

It has been oft stated that the significance of the Laguerre–Pólya class in the theory of entire functions stems from the fact that functions in this class, *and only these*, are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros. For various properties and algebraic and transcendental characterizations of functions in this class we refer the reader to Pólya and Schur [12, p. 100], [13] or [10, Kapitel II].

A real entire function $\varphi(x)$ is said to satisfy the *Laguerre inequalities* if $L_1(x) = (\varphi'(x))^2 - \varphi(x)\varphi''(x) \geq 0$ for all $x \in \mathbb{R}$. These inequalities are only necessary conditions for $\varphi(x)$ to belong to the Laguerre–Pólya class. Indeed, if, for example, $\varphi(x) = e^{x^2/2} \cos x$, then $L_1(x) = e^{x^2} \sin^2 x \geq 0$ for all $x \in \mathbb{R}$, but $\varphi \notin \mathcal{L}\text{-}\mathcal{P}$. On the other hand, as the following theorem shows, the so-called *generalized real Laguerre inequalities* (see also Theorem 3.1) are *both necessary and sufficient* for membership in the Laguerre–Pólya class.

Theorem 1.2 (Generalized Real Laguerre Inequalities [4, Theorem 2.9]).

Let φ denote a real entire function, $\varphi \not\equiv 0$. For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$, set

$$L_n(x) = L_n(x, \varphi) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} \varphi^{(j)}(x) \varphi^{(2n-j)}(x).$$

If $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, then $L_n(x) \geq 0$ for all $n \in \mathbb{N}_0$ and for all $x \in \mathbb{R}$. Conversely, suppose that

$$\varphi(x) = e^{-ax^2} \varphi_1(x), \quad a \geq 0, \quad \text{where the genus of } \varphi_1(x) \text{ is 0 or 1.}$$

If $L_n(x) \geq 0$ for all $n \in \mathbb{N}_0$ and for all $x \in \mathbb{R}$, then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$.

In accordance with the editor's suggestion to make this paper self-contained, we recall here the following standard definitions and nomenclature. Let $f(z)$ denote an entire function and let $M(r, f) = \max_{|z|=r} |f(z)|$. Then the *order* of the entire function f is defined as

$$\rho = \rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}.$$

Let $0 < |z_1| \leq |z_2| \leq |z_3| \leq \dots$ denote the absolute values of the (non-zero) zeros (if any) of $f(z)$. Then the *convergence exponent* of the zeros of f is the infimum of positive numbers α for which $\sum_{n=1}^{\infty} 1/|z_n|^\alpha$ converges. The smallest positive integer α for which this series converges is denoted by $p+1$ and p is called the *genus* of the set of zeros of f (see, for example, [1, p. 14] or [6, p. 55]). Now, if $f(z)$ is an entire function of finite order ρ with an m -fold zero at the origin, then by the Hadamard Factorization Theorem (cf. [1, p. 22]) $f(z) = z^m e^{Q(z)} \Pi(z)$, where $Q(z)$ is a polynomial of degree $q \leq \rho$ and $\Pi(z)$ is the canonical product (of genus p ; that is, p is the genus of the zeros) formed from the (non-zero) zeros of $f(z)$. Then the *genus* of $f(z)$ is defined as $\max(p, q)$.

Remarks 1.3.

Observe that $L_0(x) = \varphi^2(x)$ and to justify the appellation “generalized Laguerre expression”, note that $L_1(x) = (\varphi')^2(x) - \varphi(x)\varphi''(x)$. In addition, we remark that if the real entire function $\varphi(x)$ satisfies the generalized real Laguerre inequalities, $L_n(x) \geq 0$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$, then $\varphi(x)$ has only real zeros (cf. [4, p. 343]). For the sake of completeness, we briefly outline

the proof. Here, and in the sequel, we will need to refer to the following representation of $|\varphi(x + iy)|^2$ which can be derived by a direct calculation (see, for example, [4, 7, 11] or by using a recursion relation [3]):

$$|\varphi(x + iy)|^2 = \varphi(x + iy)\varphi(x - iy) = \sum_{n=0}^{\infty} L_n(x)y^{2n}, \quad x, y \in \mathbb{R}. \quad (1)$$

We proceed with a *reductio ad absurdum* argument. Thus, suppose that $z_0 = x_0 + iy_0$ is a non-real zero of $\varphi(x)$, where $\varphi \not\equiv 0$. Then $0 = |\varphi(z_0)|^2 = \sum_{n=0}^{\infty} L_n(x_0)y_0^{2n}$ (cf. (1)). Since $y_0 \neq 0$ and $L_n(x_0) \geq 0$, it follows that $L_n(x_0) = 0$ for all $n = 0, 1, 2, \dots$. Thus, for any $y \in \mathbb{R}$, $|\varphi(x_0 + iy)|^2 = \sum_{n=0}^{\infty} L_n(x_0)y^{2n} = 0$. But this implies that $\varphi(x) \equiv 0$, contrary to our assumption, and whence $\varphi(x)$ has only real zeros.

Remarks 1.4.

The action of the non-linear operators $\{L_n\}_{n=0}^{\infty}$ taking a real entire function $\varphi(x)$ to $L_n(x, \varphi) = L_n(x)$ is given by equation (1). We mention here, parenthetically, a couple facts about these operators. It is known that the operators L_n satisfy a simple recursive relation [3, Theorem 2.1] and that $L_n(x)$ is also a real entire function [3, Remark 2.4]. Interesting generalizations of these operators are given by Dilcher and Stolarsky [5] and Cardon [2] (see also Section 3).

2. Proof of the main result

The proof of the main result (Theorem 2.3) hinges on the Borel–Carathéodory inequality for the half-plane, $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and on a theorem of Lindelöf. For ease of reference, we commence here with the statements (and citations) of these theorems.

Theorem 2.1 (Borel–Carathéodory Inequality [9, p. 18]).

Suppose that $f: \mathbb{C}_+ \rightarrow \mathbb{C}_+$ is an analytic function. Then for $z \in \mathbb{C}_+$ and $|z| > 1$, f satisfies the inequalities

$$\frac{1}{5} |f(i)| \frac{\sin \theta}{r} < |f(z)| < 5 |f(i)| \frac{r}{\sin \theta}, \quad z = re^{i\theta}, \quad 0 < \theta < \pi.$$

As usual, $n(r)$ will denote the number of zeros of $f(z)$ in the closed disk $\{z \in \mathbb{C} : |z| \leq r\}$. Our proof will also require the determination of the type of an entire function $f(z)$ of positive integral order. This is a subtle matter, since the order can be larger than the number of zeros, $n(r)$, would indicate. In order to handle this issue, we will appeal to the following theorem of Lindelöf.

Theorem 2.2 (Lindelöf’s Theorem [1, p. 27]).

An entire function $f(z)$ of integral order $\rho \geq 1$ is of normal (finite) type if and only if (i) $n(r) = O(r^\rho)$, $r \rightarrow \infty$, and (ii) the sums

$$|S(r)| = \left| \sum_{\{z: f(z)=0, |z| \leq r, z \neq 0\}} \frac{1}{z^\rho} \right|$$

are bounded as $r \rightarrow \infty$.

The proof of Lindelöf’s Theorem in [1, p. 27] is quite involved and so we refer the interested reader to [6, Chapter 2] or to an alternate proof suggested in [8, p. 22, Problem 3].

Theorem 2.3.

Let $\varphi(x)$ denote a real entire function, $\varphi \not\equiv 0$. If $L_n(x) \geq 0$ for all $n \in \mathbb{N}_0$ and for all $x \in \mathbb{R}$, then (i) $\varphi(x)$ has only real zeros and (ii) $\varphi(x) = e^{-ax^2} \varphi_1(x)$, $a \geq 0$, where the genus of $\varphi_1(x)$ is 0 or 1. In particular, $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$.

Proof. Since the real entire function $\varphi(x)$ ($\varphi \not\equiv 0$) satisfies the generalized real Laguerre inequalities $L_n(x) \geq 0$ for all $n \in \mathbb{N}_0$ and for all $x \in \mathbb{R}$, $\varphi(x)$ has only real zeros (cf. Remarks 1.3). Thus, $\varphi(x)$ does not vanish on the simply connected domain $\mathbb{C}_+$ and whence $\varphi(x)$ has an analytic logarithm there; that is, $f(z) = \log \varphi(z)$ is analytic in $\mathbb{C}_+$. Now set $u(z) = \operatorname{Re} f(z) = \log |\varphi(z)|$ and $v(z) = \operatorname{Im} f(z)$. Then $f'(z) = \partial u(z)/\partial x + i \partial v(z)/\partial x$ and so by the Cauchy–Riemann equations

$$g(z) = -f'(z) = -\frac{\partial u(z)}{\partial x} - i \frac{\partial v(z)}{\partial x} = -\frac{\partial v(z)}{\partial y} + i \frac{\partial u(z)}{\partial y}. \quad (2)$$

Now for $z \in \mathbb{C}_+$, $e^{f(z)} = \varphi(z)$ and $e^{2\operatorname{Re} f(z)} = |\varphi(z)|^2 \geq 0$. Hence, by (1) and the assumption that $L_n(x) \geq 0$, $x \in \mathbb{R}$, $n = 0, 1, 2, \dots$, we infer the inequality

$$\frac{\partial}{\partial y} e^{2\operatorname{Re} f(z)} = 2e^{2u(z)} \frac{\partial u(z)}{\partial y} = \sum_{n=0}^{\infty} L_n(x) 2n y^{2n-1} \geq 0, \quad x \in \mathbb{R}, \quad y > 0. \quad (3)$$

Therefore, if $z = x + iy \in \mathbb{C}_+$, then, by (3), $\partial u(z)/\partial y \geq 0$ and hence, consulting (2), $g: \mathbb{C}_+ \rightarrow \mathbb{C}_+$. Thus, by the Borel–Carathéodory inequality (Theorem 2.1), there exists a positive constant C_1 , such that

$$|g(z)| \leq C_1 |z| \quad \text{for all } z \in \Omega = \{z : \operatorname{Im} z \geq |\operatorname{Re} z| + 1\}.$$

Integrating $g(z) = -f'(z)$ along a linear segment $[i, z]$, where $z \in \Omega$, we obtain the following upper estimate with a new constant $C_2 > 0$:

$$|f(z)| = \left| \int_{[i, z]} f'(w) dw + f(i) \right| \leq C_1 |z| \left(\int_{[i, z]} |dw| \right) + |f(i)| \leq C_2 |z|^2 \quad \text{for all } z \in \Omega. \quad (4)$$

Next, we fix $|z| = r \geq 1$ and note that since φ is a real entire function $|\varphi(x + iy)| = |\varphi(x - iy)|$ for all $x, y \in \mathbb{R}$. Since by assumption $L_n(x) \geq 0$ for all $n \in \mathbb{N}_0$ and for all $x \in \mathbb{R}$, it follows from (1) that for each fixed x , $|\varphi(x + iy)|$ is an increasing function of y , for $y \geq 0$. With the aid of these observations, and choosing $r \geq 1$ we deduce the following estimates:

$$\max_{|x + iy| \leq r} |\varphi(x + iy)| \leq \max_{-r \leq x \leq r} |\varphi(x + i(r+1))| = \max_{-r \leq x \leq r} e^{\operatorname{Re} f(x + i(r+1))} \leq \max_{-r \leq x \leq r} e^{|f(x + i(r+1))|} \leq e^{5C_2 r^2} \quad (\text{cf. (4)}).$$

Thus, we have established that the real entire function φ is of order $\rho(\varphi) \leq 2$ and that if $\rho(\varphi) = 2$, then φ is of normal type. Accordingly, we consider two cases. In the first place, if $\rho(\varphi) < 2$, then by the Hadamard Factorization Theorem, the canonical product of the zeros of φ has genus at most 1 (see, for example, [1, p. 22]). Secondly, suppose that $\rho(\varphi) = 2$ and that φ has an infinite number of zeros $\{x_k\}_{k=1}^{\infty}$, $x_k \neq 0$, $k \geq 1$. In this case, since φ is of normal type, we can invoke Lindelöf’s Theorem (Theorem 2.2) to conclude that the sums

$$|S(r)| = \sum_{|x_k| \leq r} \frac{1}{x_k^2}$$

are bounded as $r \rightarrow \infty$. Therefore, $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$ and the genus of the canonical product of the zeros of φ is again at most 1. Consequently, another appeal to the Hadamard Factorization Theorem shows that $\varphi(z)$ has the representation

$$\varphi(z) = c e^{az^2 + bz} z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k} \right) e^{z/x_k}, \quad (5)$$

where $a, b \in \mathbb{R}$, $x_k \in \mathbb{R} \setminus \{0\}$, m is a non-negative integer, c is a non-zero real number, and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$.

In order to complete the proof of the theorem, we need to show that in the above representation (see (5)) $a \leq 0$. The proof hinges on the fact, noted above, that $|\varphi(iy)|$ is an increasing function of y , $y > 0$. Now, a calculation shows that

$$h(y) = |\varphi(iy)| = |c||y|^m e^{-ay^2} \prod_{k=1}^{\infty} \left(1 + \frac{y^2}{x_k^2}\right)^{1/2},$$

where we adhere to the usual convention, whereby the empty product has value 1. Then for $y > 0$, logarithmic differentiation and some algebraic manipulation yield the following expression

$$H(y) = \frac{1}{y} \frac{h'(y)}{h(y)} = \frac{m}{y^2} - 2a + \sum_{k=1}^{\infty} \frac{1}{x_k^2 + y^2}. \quad (6)$$

Since $h(y)$ is a positive increasing function for $y > 0$, it follows that $H(y) \geq 0$ for all $y > 0$ (cf. the left-hand side of (6)). We next demonstrate that the assumption that $a > 0$ is untenable for it leads to a contradiction. Let $0 < \varepsilon < a/4$. Since the series $\sum_{k=1}^{\infty} 1/x_k^2$ converges, there exists a positive integer N such that for all $y > 0$,

$$\sum_{k=N+1}^{\infty} \frac{1}{x_k^2 + y^2} \leq \sum_{k=N+1}^{\infty} \frac{1}{x_k^2} < \frac{\varepsilon}{3}. \quad (7)$$

Fixing N , there exists a positive number y_0 , y_0 sufficiently large, such that for all $y \geq y_0$,

$$\frac{m}{y^2} < \frac{\varepsilon}{3} \quad \text{and} \quad \sum_{k=1}^N \frac{1}{x_k^2 + y^2} < \frac{\varepsilon}{3}. \quad (8)$$

Inequalities (7) and (8) show that, if $a > 0$, then for all $y \geq y_0$,

$$H(y) = \frac{1}{y} \frac{h'(y)}{h(y)} = \frac{m}{y^2} - 2a + \sum_{k=1}^{\infty} \frac{1}{x_k^2 + y^2} < \varepsilon - 2a < \frac{a}{4} - 2a < 0.$$

This is the desired contradiction and hence $a > 0$. Thus, $\varphi \in \mathcal{L}\text{-}\mathcal{P}$. □

Remark 2.4.

We remark that an often overlooked fact is that a real entire function of order 2 having only real zeros need not belong to the Laguerre–Pólya class. It is curious that our detailed, albeit elementary, proof that the coefficient a of z^2 (in e^{az^2}) is non-positive required the non-negativity of $L_n(x)$ only at $x = 0$. Perhaps this part of the proof of Theorem 2.3 can be made more elegant and perspicacious by noting that if the canonical product of the zeros of φ is of order 2, then (by virtue of the aforementioned argument) φ is of minimal type. However, such an approach may be less transparent if it suffers from the omission of certain relevant details.

3. Scholia

In this section, we include the proofs of some cognate results, state an open problem and formulate two conjectures, one of which is based on Cardon's recent extension of the Laguerre-type inequalities [2]. To begin with, we consider the following classical characterization of functions in the Laguerre–Pólya class via the complex Laguerre inequalities.

Theorem 3.1 (Complex Laguerre Inequalities [4, Theorem 3.8]).

Let $\varphi(x)$, $\varphi(x) \not\equiv 0$, be a real entire function. Suppose that

$$\varphi(x) = e^{-\alpha x^2} \varphi_1(x), \quad \text{where } \alpha \geq 0 \text{ and the genus of } \varphi_1(x) \text{ is 0 or 1.} \quad (9)$$

Then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ if and only if

$$|\varphi'(z)|^2 \geq \operatorname{Re}(\varphi(z)\overline{\varphi''(z)}) \quad \text{for all } z \in \mathbb{C}. \quad (10)$$

Once again, our goal is to demonstrate that Theorem 3.1 remains valid if we omit the hypothesis (9). If $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, then φ can be expressed in the form (9) and it is known [4] that φ satisfies the complex Laguerre inequalities (10). Moreover, without the hypothesis (9), we can infer from the complex Laguerre inequalities (10) that φ has only real zeros (see, for example, [7] or [4, Theorem 3.8]). In light of these remarks, we state the next theorem as follows.

Theorem 3.2.

Let $\varphi(x)$, $\varphi(x) \not\equiv 0$, be a real entire function. If the complex Laguerre inequalities (10) hold, then $\varphi \in \mathcal{L}\text{-}\mathcal{P}$.

Proof. The proof of this theorem is, *mutatis mutandis*, the same as the proof of Theorem 2.3. However, the different assumptions require appropriate modifications and therefore, even at the risk of pleonasm, we will succinctly sketch here the relevant justifications needed. As in the proof of Theorem 2.3, we set $f(z) = \log \varphi(z) = u(z) + iv(z)$, and $g(z) = -f'(z) = -v_y(z) + iu_y(z)$, where $z \in \mathbb{C}_+$. In order to follow the arguments used in the proof of Theorem 2.3, we need to justify two assertions: (i) $u_y(z) \geq 0$ when $z = x + iy \in \mathbb{C}_+$ and (ii) $|\varphi(iy)|$ is an increasing function of y , $y > 0$. In the proof of Theorem 2.3, both claims (i) and (ii) were clear by virtue of the generalized real Laguerre inequalities.

(i) Here, we consider again the expansion $|\varphi(x + iy)|^2 = \sum_{n=0}^{\infty} L_n(x) y^{2n}$, $x, y \in \mathbb{R}$. Then a calculation shows that, for each fixed $x \in \mathbb{R}$,

$$\frac{\partial^2}{\partial y^2} |\varphi(x + iy)|^2 = \sum_{n=0}^{\infty} (2n+1)(2n+2) L_{n+1}(x) y^{2n} 2[|\varphi'(z)|^2 - \operatorname{Re}(\varphi(z)\overline{\varphi''(z)})] \geq 0, \quad z \in \mathbb{C}. \quad (11)$$

Therefore, from the convexity condition (11), we deduce that, for each fixed $x \in \mathbb{R}$, $\partial|\varphi(z)|^2/\partial y = \partial e^{2u(z)}/\partial y = 2e^{2u(z)}u_y(z)$ is an increasing function of y , $y > 0$. Now for any x which is not a zero of φ , we have

$$\lim_{y \rightarrow 0^+} \frac{\partial e^{2u(z)}}{\partial y} = 2 \lim_{y \rightarrow 0^+} |\varphi(z)|^2 u_y(x + iy) = \lim_{y \rightarrow 0^+} \sum_{n=0}^{\infty} 2n L_n(x) y^{2n-1} = 0,$$

and hence $u_y(x + i0) = 0$. Since $2e^{2u(z)}u_y(z)$ is an increasing function of y , $0 = 2e^{2u(x+i0)}u_y(x + i0) \leq 2|\varphi(x + iy)|^2 u_y(x + iy)$, and $u_y(x + iy) \geq 0$, $y > 0$. On the other hand, if $\varphi(x_0) = 0$, then a continuity argument shows again that $u_y(x + iy) \geq 0$, $y > 0$. Thus, $g: \mathbb{C}_+ \rightarrow \mathbb{C}_+$ and the method of proof leading to the product representation of $\varphi(z)$ (cf. (5)) is *verbatim* the same as in the proof of Theorem 2.3.

(ii) In order to prove that $a \leq 0$ in (5), it suffices to show that $|\varphi(iy)|$ is an increasing function of y , $y > 0$. Now, we infer from the convexity condition (cf. (11)) that

$$0 < \int_0^y \frac{\partial^2}{\partial t^2} |\varphi(it)|^2 dt = \frac{\partial}{\partial y} |\varphi(iy)|^2, \quad (12)$$

where we have used the fact that $\partial/\partial y |\varphi(iy)|^2$ is an odd function. Thus, it follows from inequality (12) that $|\varphi(iy)|$ is an increasing function of y , $y > 0$. $\square$

As an immediate consequence of Theorems 2.3 and 3.2, we obtain the following corollary.

Corollary 3.3.

Let $\varphi(x)$, $\varphi(x) \not\equiv 0$, be a real entire function. Then

$$L_n(x) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} \varphi^{(j)}(x) \varphi^{(2n-j)}(x) \geq 0, \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$

if and only if $|\varphi'(z)|^2 \geq \operatorname{Re}(\varphi(z)\overline{\varphi''(z)})$ for all $z \in \mathbb{C}$.

It may be of interest to note here that the complex Laguerre expression can be also formulated in terms of two real Laguerre-type expressions. Indeed, if $\varphi(x + iy) = U(x, y) + iV(x, y)$ is a real entire function, then a calculation shows that

$$|\varphi'(z)|^2 - \operatorname{Re}(\varphi(z)\overline{\varphi''(z)}) = U_x^2 - UU_{xx} + V_x^2 - VV_{xx}.$$

The preceding analysis (cf. the convexity condition (11)) reveals that the complex Laguerre inequalities are, *a fortiori*, a direct consequence of the generalized real Laguerre inequalities. It is the converse implication that does not appear to be obvious and it motivates us to formulate, in the simplest terms, the following (ostensibly) intriguing, related problem.

Open Problem 3.4.

Let $\{u_k(x)\}_{k=0}^\infty$ be a sequence of real entire functions. For each fixed $x \in \mathbb{R}$, consider the even power series $F(y; x) = \sum_{k=0}^\infty u_k(x)y^{2k}$, $x, y \in \mathbb{R}$, with radius of convergence $R = \infty$, where $F(y; x) \not\equiv 0$. Suppose that $u_0(x) \geq 0$ for all $x \in \mathbb{R}$ and that $\partial^2 F(y; x)/\partial y^2 \geq 0$ for all $x, y \in \mathbb{R}$. If $F(y; x)$ is not a polynomial, then under what additional assumptions is it true that $u_k(x) \geq 0$ for all $k \geq 1$ and all $x \in \mathbb{R}$?

We next turn to Cardon's recent extension of the Laguerre-type inequalities [2]. In order to expedite our presentation of Cardon's main result [2, Theorem 2.1], we need to introduce the following functions. Let

$$p(z) = \sum_{k=0}^m c_k z^k = \prod_{j=1}^m (z + \alpha_j), \quad (13)$$

be an even polynomial with non-negative real coefficients $c_k \geq 0$, and suppose that $p(z)$ has at least one non-real zero. Let $\varphi(x)$, $\varphi(x) \not\equiv 0$, be a real entire function and define

$$\begin{aligned} \Phi(z, t) &= \prod_{j=1}^m \varphi(z + \alpha_j t) = \sum_{k=0}^\infty A_k(z) t^k, \quad \text{where} \\ A_k(z) &= \frac{1}{k!} \left[\frac{d^k}{dt^k} \Phi(z, t) \right]_{t=0}, \quad k \in \mathbb{N}_0. \end{aligned} \quad (14)$$

Then it is easy to see that $A_k(z)$ is also a real entire function and, as Cardon has remarked, the choice $p(z) = 1 + z^2$ produces $A_{2k+1}(z) \equiv 0$. Moreover, in this case, $A_{2k}(z) = L_k(z)$ for $k = 0, 1, 2, \dots$. Preliminaries aside, we are now in position to state the following theorem.

Theorem 3.5 (Cardon–Laguerre Inequalities [2, Theorem 2.1]).

Let $\varphi(x) = e^{-ax^2} \varphi_1(x)$, where the genus of the real entire function $\varphi_1(x)$ is 0 or 1, $\varphi_1(x) \not\equiv 0$ and $a \geq 0$. Let $p(z)$ and $A_k(z)$ denote the functions defined above (see (13) and (14)). Then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ if and only if $A_k(x) \geq 0$ for all $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$.

We expect that appropriate modifications of the techniques adopted in this paper, may lead to a proof of the following conjecture.

Conjecture 3.6.

Let $\varphi(x)$ be a real entire function, $\varphi(x) \not\equiv 0$. Let $p(z)$ and $A_k(z)$ denote the functions defined above (see (13) and (14)). If $A_k(x) \geq 0$ for all $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$, then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$.

Finally, by way of conclusion, we propose here a more challenging question. While this query has a geometric flavor, it does belong to the circle of ideas advanced in this investigation. We suspect that even a partial answer to Open Problem 3.7 could adumbrate new insights and results in the theory of distribution of zeros of entire functions.

Open Problem 3.7.

Let $S_\tau = \{z : |\operatorname{Im} z| \leq \tau, \tau \geq 0\}$ denote the strip of width 2τ located symmetrically about the real axis. Let $f(z)$ be a real entire function. Under what additional assumptions is it possible to characterize non-trivial regions $\Omega \subsetneq \mathbb{C}$ such that if the complex Laguerre inequalities are restricted to Ω ; that is, $|f'(z)|^2 \geq \operatorname{Re}(f(z)\overline{f''(z)})$, $z \in \Omega$, then all the zeros of $f(z)$ lie in the strip S_τ .

Acknowledgements

The second author is deeply grateful to her daughter, Liudmyla Kadets, who always wins her wagers.

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