Interior barriers for robust positivity programs toward the Riemann hypothesis

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Abstract

Rodgers and Tao's theorem that the de Bruijn–Newman constant is nonnegative supplies an explicit sequence of real entire functions $H_{-1/n}$, each with a nonreal zero, converging analytically to the Riemann function H_0. We isolate the exact topological consequence: if a certificate map sends those counterexamples to the target certificate, if its accepted region is open at the target, and if the same acceptance implication forces real-rootedness on the target and on every counterexample, then the target cannot be accepted. The proof is elementary; the substance is the operational membership test and its application to complete positivity architectures.

We prove the corresponding cone-interior dichotomies for infinite scalar hierarchies and self-adjoint operators. Weak coordinate topologies have empty positive interior, whereas weighted sup and operator-norm topologies have interior exactly at a uniform positive floor or a coercive bound. We then audit seven primary-source programs: de Bruijn–Newman heat flow, Jensen polynomials, generalized Laguerre and Turan inequalities, Weil forms, de Branges spaces, canonical systems, and Polya-frequency criteria. The Li and Nyman–Beurling criteria serve as boundary controls. None of the cited papers is itself a member of the obstruction class. What is excluded is a precisely specified future completion combining a fixed topology, regular transport along an explicit false family, an open uniform reserve, and common sufficiency. Semidefinite, zero-infimum, singular, changing-domain, endpoint-arithmetic, and exact boundary mechanisms remain outside the barrier. The abstract theorem and its coercive and finite-test specializations are checked in Lean.

Keywords. Riemann hypothesis; de Bruijn–Newman constant; positivity criteria; Laguerre–Polya class; Weil quadratic form; total positivity; natural barriers; formal verification.
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1 Introduction

Many criteria equivalent to the Riemann hypothesis are positivity statements: hyperbolicity of every Jensen polynomial, nonnegativity of every generalized Laguerre expression, positivity of the Weil quadratic form, positive-semidefinite canonical Hamiltonians, or total positivity of a reciprocal transform. Their finite or localized instances often admit strict estimates. It is therefore natural to ask whether a complete proof can be obtained by making those estimates uniform and stable.

This paper identifies a precise obstruction to that final move. It does *not* say that positivity methods are impossible, nor that the papers audited below made an invalid argument. It says that a complete certificate cannot simultaneously have all four properties:

1. an explicit sequence of false objects approaching the target;

2. convergence of their certificate values in one fixed topology;
3. an open accepted neighborhood at the target; and
4. one acceptance theorem valid for both target and counterexamples.

The topological proof is one line. The RH-specific force comes from a canonical, unconditional false sequence. Writing

$$H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du, \tag{1}$$

we normalize the real Riemann function as $\Xi(z) = H_0(z)$. There is a finite de Bruijn–Newman constant Λ such that H_t has only real zeros exactly when $t \geq \Lambda$. Rodgers and Tao proved $\Lambda \geq 0$ [12, Theorem 1]. Consequently,

$$F_0 = H_0, \qquad F_n = H_{-1/n} \quad (n \geq 1) \tag{2}$$

is an explicit sequence in which every F_n, $n \geq 1$, has a nonreal zero.

The contribution has three parts. First, Section 2 states the exact obstruction in topological and metric form. Second, Section 3 identifies what "open positivity" means for the complete scalar and operator cones that occur in RH programs. Third, Sections 4–5 give a primary-source membership protocol and apply it program by program. The audit deliberately separates an actual paper's theorem from a hypothetical robust completion of its architecture.

The conclusion is prospective and falsifiable. In the audited architectures, a successful proof must do at least one of the following: remain on a sharp boundary, lose uniform regularity along the canonical false family, use a target-specific mechanism unavailable on that family, change domain or normalization essentially, or quantitatively separate the false family from Ξ. Seeking a uniform interior reserve while preserving common backward-heat regularity is not the missing step.

2 The accumulating-counterexample barrier

Let X be a class of objects, $\mathsf{Good} \subseteq X$ the desired property, $x_0 \in X$ the target, and $x_n \in X$ explicit counterexamples. A certificate is a map $\mathcal{D} : X \to Y$ into a topological space Y, together with an accepted set $U \subseteq Y$. Only the restriction of these data to

$$\mathcal{S} = \{x_0\} \cup \{x_n : n \geq 1\}$$

matters.

Definition 2.1 (AIC membership)**.** An argument is an *accumulating-counterexample interior certificate* argument, abbreviated AIC, if an outside reader can verify:

M1: explicit false family. $x_n \notin \mathsf{Good}$ for every $n \geq 1$.

M2: certificate accumulation. $\mathcal{D}(x_n) \to \mathcal{D}(x_0)$ in Y.

M3: open acceptance. U is open and $\mathcal{D}(x_0) \in U$.

M4: common sufficiency. For every $x \in \mathcal{S}$, $\mathcal{D}(x) \in U$ implies $x \in \mathsf{Good}$.

The carrier Y, topology, normalization, index set, and map $\mathcal{D}$ must be fixed as part of the argument before the terminal estimate at x_0. An endpoint-tailored or discrete topology is a nonmember rather than a counterexample to the theorem.

Theorem 2.2 (Interior barrier)**.** *Conditions M1–M4 are inconsistent. More precisely, under M1, M2, and M4, the point $\mathcal{D}(x_0)$ does not lie in the interior of any accepted set U.*

Proof. If $\mathcal{D}(x_0) \in \operatorname{int} U$, then $\operatorname{int} U$ is an open neighborhood of $\mathcal{D}(x_0)$. Convergence gives N such that $\mathcal{D}(x_n) \in \operatorname{int} U \subseteq U$ for every $n \geq N$. M4 then gives $x_n \in \mathsf{Good}$, contradicting M1. Taking U itself open proves the M1–M4 formulation. □

Remark 2.3 (Why state an elementary lemma?)**.** Theorem 2.2 is elementary and should not be advertised as deep topology. Its use is diagnostic. Rodgers–Tao supplies a canonical false family; complete positivity programs supply recognizable certificate cones; and the M1–M4 checklist prevents pointwise continuity, finite strictness, or target-only sufficiency from being silently upgraded to a robust RH proof.

Corollary 2.4 (Metric form)**.** *Let (Y, d) be a metric space. Suppose $d(\mathcal{D}(x_n), \mathcal{D}(x_0)) \to 0$ and acceptance is sufficient on $\mathcal{S}$, while every x_n is bad. Then there is no $r > 0$ for which $B(\mathcal{D}(x_0), r) \subseteq U$.*

Proof. Choose n with $d(\mathcal{D}(x_n), \mathcal{D}(x_0)) < r$ and apply M4. □

Corollary 2.5 (Canonical RH specialization)**.** *Take $x_0 = H_0$, $x_n = H_{-1/n}$, and let* Good *mean "has only real zeros." Under M2 and M4, H_0 cannot satisfy M3.*

Proof. Rodgers–Tao's $\Lambda \geq 0$ theorem makes every $H_{-1/n}$ bad, so Theorem 2.2 applies. □

The same statement can be phrased directly on the heat path. If a certificate at $t = 0$ persists on some interval $(-\varepsilon, 0]$, and if the certificate implies real-rootedness for each H_t there, then it contradicts $\Lambda \geq 0$. The sequence form is stronger because it applies to any explicit false family and any certificate topology, not only to interval persistence.

3 Interiors of positivity cones

The abstract theorem becomes informative only after the relevant notion of interior is made explicit. Complete positivity criteria usually face a dichotomy: weak topologies make the positive cone have empty interior, while strong uniform topologies make interior equivalent to a positive reserve.

3.1 Infinite scalar hierarchies

Let A be an infinite index set and

$$C_+ = \{q \in \mathbb{R}^A : q_\alpha \geq 0 \text{ for every } \alpha \in A\}.$$

Proposition 3.1 (Product and weighted-sup dichotomy)**.**

1. *In the product topology on $\mathbb{R}^A$, C_+ has empty interior.*

2. *Fix weights $w_\alpha > 0$ and let*

$$\ell^\infty_w(A) = \left\{ q : \|q\|_{\infty,w} := \sup_{\alpha\in A} \frac{|q_\alpha|}{w_\alpha} < \infty \right\}.$$

Write $C_{+,w} = C_+ \cap \ell^\infty_w(A)$. Then

$$\operatorname{int}_{\ell^\infty_w(A)} C_{+,w} = \left\{ q \in \ell^\infty_w(A) : \inf_{\alpha\in A} \frac{q_\alpha}{w_\alpha} > 0 \right\}.$$

Proof. A basic product neighborhood controls only finitely many coordinates. Choose an uncontrolled coordinate and make it negative; hence no point of C_+ is interior.

For the weighted sup norm, if $\delta = \inf_\alpha q_\alpha/w_\alpha > 0$, then every p with $\|p - q\|_{\infty,w} < \delta$ is positive coordinatewise. Conversely, if the infimum is zero, then for every $r > 0$ choose α with $q_\alpha/w_\alpha < r/2$ and alter only that coordinate by $-3rw_\alpha/4$. The perturbation has norm $3r/4 < r$ and leaves the cone. □

Corollary 3.2 (Uniform hierarchy barrier)**.** *Suppose $q_\alpha(t)$ is a complete sufficient hierarchy for real-rootedness of H_t, with $(q_\alpha(t))_{\alpha\in A} \in \ell^\infty_w(A)$ for t near zero. If, for fixed a-priori weights,*

$$\sup_\alpha \frac{|q_\alpha(t) - q_\alpha(0)|}{w_\alpha} \longrightarrow 0 \quad (t \uparrow 0)$$

and $q_\alpha(0) \geq \delta w_\alpha$ for every α and some $\delta > 0$, then a contradiction follows.

Proof. The endpoint lies in the weighted-sup interior of the positive cone, and the negative-heat certificates converge to it. Apply Corollary 2.5. □

This also exposes a normalization tradeoff. If positive endpoint coordinates are rescaled after inspection by $w_\alpha = q_\alpha(0)$, the margin becomes one, but M2 becomes the relative uniform estimate

$$\sup_\alpha \frac{|q_\alpha(t) - q_\alpha(0)|}{q_\alpha(0)} \longrightarrow 0.$$

The barrier forbids that estimate together with common sufficiency. Moreover, endpoint-dependent rescaling violates the a-priori normalization clause in Definition 2.1.

3.2 Quadratic forms and operators

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $B(\mathcal{H})_{\mathrm{sa}}$ the bounded self-adjoint operators.

Proposition 3.3 (Operator-cone dichotomy)**.** *For the positive cone*

$$\mathcal{P} = \{A \in B(\mathcal{H})_{\mathrm{sa}} : A \geq 0\},$$

one has:

1. *in operator norm,*
$$\operatorname{int}\mathcal{P} = \{A : A \geq \delta I \text{ for some } \delta > 0\};$$
2. *in the strong or weak operator topology, $\mathcal{P}$ has empty interior.*

Proof. If $A \geq \delta I$, every self-adjoint B with $\|B - A\| < \delta$ is positive. Conversely, if no such δ exists, choose unit vectors u_n with $\langle Au_n, u_n \rangle \to 0$; arbitrarily small negative rank-one perturbations leave the cone.

A basic strong neighborhood controls A on finitely many vectors. Choose a unit vector u orthogonal to their span and replace A by $A - M(u \otimes u)$, with $M > \langle Au, u \rangle$. The controlled vectors are unchanged but positivity fails. The same perturbation works for a basic weak neighborhood after choosing u orthogonal to both finite lists of test vectors appearing in its matrix coefficients. □

Corollary 3.4 (Coercive-form barrier)**.** *Let Q_t be quadratic forms on one normed space V, and suppose nonnegativity of Q_t implies real-rootedness of H_t. If*

$$Q_0(v) \geq \delta \|v\|^2$$

for some $\delta > 0$, and for every $\eta > 0$ there is $\varepsilon > 0$ such that

$$|Q_t(v) - Q_0(v)| \leq \eta \|v\|^2 \quad (-\varepsilon < t \leq 0,\ v \in V),$$

then a contradiction follows.

Proof. Use $\eta = \delta/2$. Then $Q_t(v) \geq (\delta/2)\|v\|^2$ for all sufficiently small negative t, contrary to Corollary 2.5. □

The norm scaling in Corollary 3.4 is essential. A bound $Q_0(v) \geq \delta$ for every v is vacuous at $v = 0$; a continuity estimate independent of $\|v\|$ does not describe form-norm continuity.

3.3 Finite strict tests

Corollary 3.5 (Finite-test barrier)**.** *Let I be finite. Suppose every $q_i(t)$ is continuous at 0, $q_i(0) > 0$ for all $i \in I$, and simultaneous nonnegativity of the $q_i(t)$ implies real-rootedness of H_t. Then these assumptions are inconsistent.*

Proof. Finiteness and strict positivity give a common negative-time neighborhood on which every $q_i(t) > 0$, contradicting $\Lambda \geq 0$. □

The corollary does not reject finite computations stated at finite scope. It rejects only the additional assertion that finitely many strict continuous tests are sufficient for the entire function.

4 Membership protocol

The source audit uses two separate questions.

> **Source question.** Does the cited paper itself supply M1–M4?
>
> **Completion question.** What exact additional statements would turn a recognizable certificate architecture into an AIC member?

Outside-reader worksheet. For any paper or proposed proof, record the following data without substituting words such as "stable," "strict," or "uniform" for an estimate.

1. State the target x_0, the desired property Good, and an explicit sequence x_n with a cited proof that every x_n is bad (M1).
2. Write the certificate map $\mathcal{D}$, its domain, index set, normalization, carrier Y, and topology. These must be fixed independently of the terminal margin claimed at x_0.
3. Prove $\mathcal{D}(x_n) \to \mathcal{D}(x_0)$ in that exact topology, not only coordinatewise in each fixed finite truncation (M2).
4. Identify the accepted set U and exhibit an open neighborhood of $\mathcal{D}(x_0)$ contained in U, equivalently an explicit reserve in a metric model (M3).
5. Quote or prove one same-object implication $\mathcal{D}(x) \in U \Rightarrow x \in \mathsf{Good}$ for x_0 and every named x_n, checking all domain and transform hypotheses there (M4).
6. Stop at the first failed item and report "nonmember" together with that escape hatch. If all four premises are verified, the purported completion is inconsistent by Theorem 2.2; one of its claimed premises must be false.

No paper in Table 1 supplies all four premises. Thus none is refuted by Theorem 2.2. Each "ruled out" verdict below has the conditional form: *if a future argument adds every listed admission clause, then the theorem applies*.

The seven rows are architecture representatives, not a claim that the bibliography exhausts every equivalent formulation of RH. An omitted paper receives no verdict by analogy: an outside reader must run the same worksheet. Any omitted argument that passes all four items is covered by the abstract theorem; one that does not pass is outside the class.

Table 1. Premises supplied by the primary sources. "General M4" means that a source theorem gives sufficiency for a general real-entire or transform class once its hypotheses are verified.

Program	Source-supplied M1–M4	Actual-source verdict
de Bruijn–Newman heat	M1 yes; M2–M4 absent for an argument-chosen certificate	Theorem input, not a member
Jensen polynomials	General M4 via Polya–Schur; M1–M3 absent	Nonmember
Generalized Laguerre–Turan	General M4 via Theorem 2.3; M1–M3 absent	Nonmember
Weil–Bombieri forms	Target-only M4; no false family or heat form; no norm-open reserve	Nonmember
de Branges positivity	Proposed M3 fails at the target; no false-family transport	Refuted in its own source
Canonical systems	No M1–M4 heat package; ω is not heat time	Nonmember
Polya frequency	General M4 is conditional on transform hypotheses; M1–M3 absent	Nonmember

Three rules prevent false membership.

Topology and normalization are argument data. One may not choose a metric after seeing the endpoint estimates, isolate Ξ by decree, or divide each coordinate by its observed endpoint value and then call the result robust. Such constructions either fail the a-priori clause or move the burden to M2.

Pointwise continuity is not M2. Continuity of each fixed Jensen polynomial, Laguerre expression, minor, or matrix entry does not imply convergence in the complete uniform topology that gives the positive cone an interior.

Sufficiency must follow the same objects. A zeta-specific prime identity at $t = 0$ does not imply real-rootedness of H_t. A reciprocal transform or canonical system defined only at the endpoint cannot be transported formally. Failure of a common domain, transform, or sufficiency theorem is a genuine escape hatch, not a technical omission.

5 Primary-source dossiers

5.1 de Bruijn–Newman heat flow and universal factors

De Bruijn's Theorem 13 contracts a strip of zeros under multiplication of the Fourier kernel by a Gaussian factor, and Theorem 14 places this in the universal-factor framework [7, pp. 204–205]. Rodgers and Tao state the threshold characterization and prove $\Lambda \geq 0$ [12, pp. 2–3].

Actual source. M1 is supplied by (2). The papers do not choose an arbitrary certificate $\mathcal{D}$, prove $\mathcal{D}(H_{-1/n}) \to \mathcal{D}(H_0)$, or claim an open RH-sufficient endpoint certificate. They are theorem inputs, not AIC members.

Precisely excluded completion. Any argument in this deformation architecture is ruled out if it defines one fixed certificate topology on $H_0, H_{-1}, H_{-1/2}, \ldots$, proves certificate convergence, accepts H_0 in an open set, and proves acceptance sufficient for real-rootedness on the same family.

What survives. Positive-time upper bounds for Λ, exact identities at $t = 0$, one-sided limits that yield no negative interval, and certificates singular under backward heat.

5.2 Jensen-polynomial hyperbolicity

O'Sullivan's Theorem 1.1 states that RH is equivalent to hyperbolicity of every $J^{d,n}$, $d \geq 1, n \geq 0$ [11, p. 2]. His Theorem 3.1 records the general Polya–Schur theorem: all Jensen polynomials of a real formal power series are hyperbolic exactly when it converges to an entire function in the Laguerre–Polya class [11, p. 6]. Thus the complete hierarchy has a general sufficiency direction, not merely a zeta-specific one.

The heat embedding can be made exact. Since H_t is even, the power-series definition

$$\Theta_t(w) = H_t(2i\sqrt{w}), \qquad H_t(z) = \Theta_t(-z^2/4),$$

is single-valued and entire. The displayed Riemann kernel $\Phi(u)$ is positive for $u \geq 0$, so Θ_t has positive Taylor coefficients for $t \leq 0$. If every Jensen polynomial of Θ_t is hyperbolic, Polya–Schur gives $\Theta_t \in \mathcal{LP}$; its positive coefficients exclude positive real zeros. Hence all zeros of Θ_t are nonpositive, and the second identity forces every zero of H_t to be real. This verifies M4 on the same heat family rather than borrowing a target-only zeta implication.

Let $\mathcal{P}_d$ be the coefficient space of monic degree-d polynomials and $\mathcal{H}_d \subseteq \mathcal{P}_d$ its closed hyperbolic set. The a-priori normalization is assumed to place each Jensen polynomial in this fixed chart and to preserve hyperbolicity. For $d \geq 2$ and an accepted polynomial p, the robust reserve is

$$r_d(p) = \operatorname{dist}(p, \mathcal{P}_d \setminus \mathcal{H}_d).$$

The complete certificate consists of a-priori normalized polynomials $N_{d,n} J_F^{d,n}$. In a weighted sup metric, open acceptance means a positive infimum of the corresponding distances to the complements. The distance is a reserve on accepted points; it should not be confused with a globally signed certificate, since distance to a set is always nonnegative.

Actual source. The paper supplies neither the negative-heat family nor a uniform coefficient modulus over all (d, n), and it proves no positive complete reserve. Griffin, Ono, Rolen, and Zagier prove in Theorem 1 that for every fixed d, $J^{d,n}$ is hyperbolic for all sufficiently large n [10, p. 2]. This is a fixed-degree tail theorem, not M3. Farmer further shows that fixed-degree hyperbolicity can persist for off-axis entire functions and therefore loses the information needed for RH [8].

Precisely excluded completion. Fix $N_{d,n}$ and a complete coefficient metric independently of the endpoint; apply the construction to every $H_{-1/k}$; prove convergence in that complete metric; and prove a positive uniform distance from the nonhyperbolic complements. These additions make the program an AIC member and are inconsistent.

What survives. Every fixed degree, every finite index range, and complete hyperbolicity whose normalized distances have infimum zero.

5.3 Generalized Laguerre and Turan inequalities

Csordas, Norfolk, and Varga prove the classical Turan inequalities in Theorem 2.5 [5, p. 528]. Their introduction identifies these inequalities as necessary for the relevant real-rootedness statement; failure would disprove RH, but validity does not prove it. Csordas and Vishnyakova define

$$L_m(x; F) = \sum_{j=0}^{2m} \frac{(-1)^{j+m}}{(2m)!} \binom{2m}{j} F^{(j)}(x) F^{(2m-j)}(x) \tag{3}$$

and prove in Theorem 2.3 that $L_m(x; F) \geq 0$ for all $m \geq 0$ and $x \in \mathbb{R}$ forces a nonzero real entire F into the Laguerre–Polya class [6, pp. 1645–1647]. Their Remarks 1.3 give the direct real-zero implication through

$$|F(x + iy)|^2 = \sum_{m=0}^{\infty} L_m(x; F) y^{2m}.$$

Certificate discipline. The automatic coordinate $L_0(x;F) = F(x)^2$ vanishes at every real zero and cannot carry a strict margin. It may be omitted from a nonnegativity certificate without weakening sufficiency. A robust completion would instead fix weights $w_{m,x} > 0$ for $m \geq 1$ and use a topology controlling $L_m(x;F)/w_{m,x}$ uniformly over order and point.

Actual source. The general sufficiency theorem supplies M4 once the hierarchy is defined, but no source supplies the heat false family, a uniform backward modulus, or a positive normalized all-order/all-point floor. Unnormalized expressions may also decay as $|x| \to \infty$.

Precisely excluded completion. A fixed normalized hierarchy on all $H_{-1/k}$, a uniform heat modulus over m and x, and a positive complete floor at H_0.

What survives. The proved Turan inequalities, fixed Laguerre order, compact x ranges, pointwise strictness, and complete hierarchies with zero normalized infimum.

5.4 Weil and Bombieri quadratic forms

Bombieri states Weil's criterion as positivity of a quadratic functional and gives a necessary-and-sufficient formulation in Theorem 1 [2, pp. 191–192]. He studies variational problems on functions with prescribed support and finite matrix truncations. His theorem is pointwise strict for each nonzero test function under RH, but it does not give a positive lower bound for the Rayleigh quotient in any fixed complete norm. Pointwise positive definiteness is therefore not coercive interior. Suzuki restates

$$Q_W(v) \geq 0 \quad \text{for every } v \in C_c^\infty(\mathbb{R}) \quad \Longleftrightarrow \quad \text{RH}$$

and proves continuity of the lowest localized eigenvalue $a \mapsto \lambda_a$ in Theorem 1.3 [14, pp. 1,5]. If RH fails, $\lambda_a < 0$ for some a, whereas it is positive for small a; continuity therefore produces a zero crossing.

Actual source. The forms are arithmetic zeta forms. Neither paper defines Q_t for the functions H_t, proves a common real-rootedness implication, or asserts a global coercive floor. Pointwise strict positivity, semidefinite positivity, and support-dependent lowest eigenvalues are boundary data rather than open operator-norm acceptance unless a uniform Rayleigh floor is proved.

Precisely excluded completion. Forms Q_t on one fixed form domain V, with positivity of Q_t implying real-rootedness of the same H_t, convergence in form norm, and $Q_0(v) \geq \delta \|v\|_V^2$. Corollary 3.4 rules out exactly this completion.

What survives. Semidefinite forms, explicit null sequences, support-dependent floors, changing domains, distributional limits, and prime-side identities available only at $t = 0$. Suzuki's zero crossing is an exemplar of a surviving boundary mechanism.

5.5 de Branges-space positivity

Conrey and Li's Theorem 1 gives hypotheses under which

$$\mathrm{Re}\langle F(z+i), F(z)\rangle_{H(E)} \geq 0$$

for every admissible F forces the zeros of E onto the target line [4, p. 930]. They then test $E(z) = \xi(1 - iz)$. Equation (3.2), evaluated at the 34th zeta zero, has the wrong sign, and the authors conclude that the zeta-associated space does not satisfy condition (3.1) [4, p. 934].

Actual source. The proposed endpoint positivity already fails. It is not an AIC member and is not refuted by Theorem 2.2; it is refuted inside its own primary source.

Precisely excluded replacement. A different de Branges condition would enter the class only after it is true at H_0, is defined on $H_{-1/n}$ using one common domain, has same-object real-rootedness sufficiency, converges in form norm, and has a coercive interior reserve.

What survives. Different de Branges spaces, semidefinite kernels, changing domains, and singular endpoint constructions.

5.6 Canonical-system Hamiltonians

Suzuki defines shifted functions A^{ω} and proves in Proposition 1.1 that RH is equivalent to real-rootedness of A^{ω} for every $\omega > 0$ [13, p. 1]. Theorem 2.3 constructs the associated canonical system unconditionally for $\omega > 1$; the paper explains that an extension to all $\omega > 0$ would yield a criterion through positive-semidefinite Hamiltonian matrices [13, pp. 2,23–24].

Actual source. The parameter ω is a shift in the completed zeta function, not de Bruijn–Newman heat time. The construction is semidefinite, has parameter-dependent analytic difficulties, and is not defined on $H_{-1/n}$. M1–M4 are therefore absent.

Precisely excluded completion. A backward-heat canonical system on one common measurable domain and normalization, same-object real-rootedness sufficiency, convergence in a positivity-preserving norm, and uniform positive definiteness.

What survives. Positive-semidefinite Hamiltonians, ω-singular limits, changing domains, and constructions with no heat extension.

5.7 Polya-frequency and total positivity

Gröchenig's Theorem 4 states that RH holds exactly when

$$\Lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ix\tau}}{\xi(1/2+\tau)}\, d\tau \tag{4}$$

is a Polya-frequency function [9, p. 4]. This follows from Schoenberg's general transform characterization. The paper also states that even the order-one positivity, equivalently positive definiteness of the reciprocal function, is not known [9, p. 5].

Certificate discipline. Raw translation minors vanish when nodes coalesce. Thus a purported positive floor over all raw node configurations is impossible. Any robust all-minor certificate must specify in advance either separated configurations or a confluent normalization and must include every limiting configuration needed for PF sufficiency.

Actual source. The reciprocal transform in (4) is proved well-defined for the endpoint used in the paper. The source does not define corresponding transforms for $H_{-1/n}$, verify integrability there, prove a uniform minor modulus, or produce an open PF reserve.

Precisely excluded completion. Define source-compatible reciprocal transforms throughout the canonical false family; verify Schoenberg's hypotheses there; fix a complete minor normalization and topology; and prove both certificate accumulation and open PF acceptance.

What survives. Fixed PF order, rank-dependent reserves tending to zero, coalescing-node boundary behavior, and singular or nonexistent reciprocal transforms.

6 Boundary controls and escape hatches

6.1 Li coefficients

Bombieri and Lagarias prove a general multiset criterion in Theorem 1 and its critical-line corollary in Corollary 1 [3, pp. 276–278]. Under explicit summability and symmetry hypotheses, critical-line location is equivalent to nonnegativity of every generalized Li sum

$$\lambda_n(R) = \sum_{\rho \in R} \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right].$$

This is an all-index boundary criterion. The source contains no canonical heat deformation of R, no fixed normalization producing an open reserve, and no uniform modulus. A future source-matched false multiset family satisfying those additions would fall under Theorem 2.2; endpoint arithmetic formulas and zero complete infimum do not.

6.2 Nyman–Beurling

Báez-Duarte's Theorem 1.1 states RH as membership of a fixed vector in the closure of the natural Beurling span [1, p. 1]. The terminal condition is distance zero to a closure. It is therefore an exact boundary statement, not an open positivity certificate. This control prevents "Hilbert space" or "stable approximation" from being used as proxy tests for AIC membership.

6.3 Recorded surviving mechanisms

At the theorem level the alternatives are exhaustive: an argument evades the contradiction only by failing M1, M2, M3, or M4. With the Rodgers–Tao false family fixing M1, the recurrent concrete failures of M2–M4 in the audited sources include:

1. all-index positivity with zero complete infimum;

2. semidefinite forms, null sequences, and zero eigenvalue crossings;
3. rank-, degree-, point-, support-, or height-dependent margins;
4. heat-singular transforms and changing spaces, domains, or normalizations;
5. arithmetic identities defined only for the zeta endpoint;
6. exact closure, density, and limiting criteria;
7. certificates that quantitatively separate every named false family from Ξ;
8. positive-time de Bruijn–Newman estimates; or
9. finite results stated only at finite scope.

Conversely, saying that a program is "robust" does not place it in the class. The outside reader must exhibit the exact false family, certificate topology, open accepted set, and common sufficiency theorem. At the first missing premise, the verdict narrows to nonmembership.

7 Hostile checks

The following objections are load-bearing.

The theorem is tautological. Yes: openness plus convergence eventually accepts the sequence. The paper's claim is the exact RH specialization, cone-interior identification, and source classification, not a difficult topological lemma.

A topology can be chosen to isolate the target. Yes, but that makes M2 false or violates the a-priori topology clause. The theorem does not outlaw unnatural certificates; it classifies them as nonmembers.

Every fixed coordinate is continuous. That proves only product-topology convergence. By Proposition 3.1, the positive cone then has empty interior. A topology with positive interior requires the complete uniform estimate that triggers the barrier.

Strict positivity at every index should be enough. No. The sequence $q_n = 1/n$ is strictly positive but lies on the boundary of the positive cone in ℓ^∞. The operator $\mathrm{diag}(1, 1/2, 1/3, \ldots)$ is positive and injective but not coercive.

The false family may leave the certificate domain. Correct. This defeats M2 or M4. It is a central surviving mechanism for reciprocal transforms, arithmetic forms, de Branges spaces, and canonical systems.

A criterion may imply RH only through zeta arithmetic. Correct. Endpoint-only prime identities are outside M4 and therefore survive.

Finite strict computations are useful. Correct. Corollary 3.5 applies only if those finite tests are also claimed sufficient for real-rootedness of the complete function.

No cited paper is a member. Correct. The result is prospective: it excludes a common proposed completion move, not the papers' valid finite and boundary theorems.

8 Formal verification and reproducibility

The supplementary Lean file `HeatBoundary.lean` checks:

1. direct backward-persistence and metric heat-path barriers;
2. metric, open-set, and arbitrary-interior accumulating-counterexample theorems;
3. a uniform scalar-hierarchy specialization;
4. the norm-scaled common-domain coercive-form specialization; and
5. the finite strict continuous-test corollary.

The topological theorem's sufficiency premise is restricted exactly to

$$\{x_0\} \cup \mathrm{range}(x_n),$$

not to every object of the ambient type. In Lean this scope is `Set.insert target (Set.range counterexample)`, matching M4 premise-for-premise.

The reproducibility bundle also contains:

- SHA-256 hashes for every primary-source PDF;
- OCR or text extraction with theorem anchors;
- a machine-readable dossier giving source and completion verdicts;
- a verifier that compiles Lean, validates source hashes and anchors, checks the exact seven claimed program dossiers and two boundary controls, builds this manuscript, and inspects the resulting PDF text.

The analytic statement $\Lambda \geq 0$ is cited rather than formalized. The Lean development proves the complete deduction from the explicit bad-family premise.

9 Conclusion

The de Bruijn–Newman theorem turns backward heat into a universal stress test for robust RH certificates. If a complete positivity architecture gives Ξ an open reserve and transports that reserve continuously to the canonical negative-heat functions, common sufficiency certifies a known false object. The obstruction is topological, but its practical content is quantitative: product topologies have no positive interior, while the natural strong topologies demand exactly the uniform floor or coercivity that cannot persist.

The source audit sets a narrow boundary around this claim. The cited heat, Jensen, Laguerre–Turan, Weil, de Branges, canonical, and total-positivity sources are not members of the obstruction class. Their semidefinite, finite, localized, singular, and exact endpoint results remain valid. What future work should abandon is the search for a terminal *uniform interior reserve* that is regular on an explicit false family and sufficient there.

The resulting fork is constructive. A viable program must reach RH on a boundary, explain a controlled singularity, retain genuinely zeta-specific arithmetic, change its domain essentially, or build a certificate that separates the canonical counterexamples. Those are not loopholes around the theorem; they are the remaining mathematical directions.

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